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Mathematical Reviews

Vol. 5, No. 8 SEPTEMBER, 1944 Pages 197-224

FOUNDATIONS

Hailperin, Theodore. A set of axioms for logic. J. Symbolic Logic **9**, 1–19 (1944).

The author presents an axiomatic simplification of the logical system given by W. V. Quine [Amer. Math. Monthly **44**, 70–80 (1937)]. Quine's formulation of his system involved the notion of stratified formulas, which were defined in a formal way but, intuitively speaking, can be characterized simply as formulas which would be meaningful according to the theory of types. Quine used this notion of stratification in the following axiom schema. If ϕ is stratified and does not contain x, then

$$(Ex)(y)[yex \equiv \phi]$$

is a theorem. The author formulates nine rather simple axioms, no one of which involves the notion of stratification, and gives a detailed proof that the system obtained by replacing the above axiom schema by these nine axioms is equivalent to Quine's original system (that is, the same formulas are provable in both systems). Thus it is possible to present Quine's system without making use of the notion of stratification. This result may in the future be found helpful in the investigation of meta-logical problems regarding Quine's system. *J. C. C. McKinsey.*

Novikoff, P. S. On the consistency of certain logical calculus. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 231–261 (1943). (English. Russian summary) [MF 9586]

This paper is concerned with a certain (nonconstructive) system of logic which is obtained by introducing infinite products and sums into the ordinary (two-valued) propositional calculus. The author defines the class of "true" formulas of this system by a set of axioms very similar to those used to define provable formulas in the restricted function calculus. The adjectives "provable" and "valid" are applied to formulas without explanation. The author also defines a class of formulas called "regular" and gives what purports to be a proof that every "true" formula is "regular." From this he concludes that (1) the system is consistent and (2) if a formula of the form $\sum_{i=1}^{\omega} F_i$ is "true," where for every i either F_i or $\bar{F}_i$ is "true," then one can find an N such that $\sum_{i=1}^{N} F_i$ is also "true." These proofs are carried out in an intuitionistic metamathematics. It is asserted at the beginning of the paper that "from such a point of view it appears to be possible to prove the consistency of Russell's formalism consisting of a simple theory of types and the infinity axiom" (and in the abstract is added "and Zermelo's hypothesis"), but nothing further is said regarding this point.

On account of the faulty English translation, the carelessness in formulating definitions and the multitude of typographical errors, this reviewer has been unable to understand many of the proofs. *J. C. C. McKinsey.*

Bochvar, D. A. On the consistency of a three-valued logical calculus. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 353–369 (1943). (Russian. English summary) [MF 10229]

The present paper contains an investigation of a three-valued logical calculus (the Σ system) previously described by the author [Rec. Math. [Mat. Sbornik] N.S. **4(46)**, 287–308 (1938)].

A constructive consistency proof is given for a part of this calculus rendering the results previously published concerning the Russell paradox. A method for a non-constructive consistency proof for the complete calculus is briefly indicated. *Author's summary.*

Löwig, Henry. On the importance of the relation

$$[(A, B), (A, C)] < [(B, C), (C, A), (A, B)]$$

between three elements of a structure. Ann. of Math. (2) **44**, 573–579 (1943). [MF 9398]

Let a,b,c be any elements of a lattice L and let $x = a \cap [(b \cap c) \cup (c \cap a) \cup (a \cap b)]$, $y = (b \cap c) \cup (b \cap a)$, $z = (c \cap a) \cup (c \cap b)$. Dedekind showed that $(x \cap y) \cup (x \cap z) = (a \cap b) \cup (a \cap c) = x$ if L is modular. Löwig shows that $(x \cap y) \cup (x \cap z) = (u \cap b) \cup (a \cap c) \leqq x$ in any case, and that x,y,z generate a sublattice S of at most nine elements. Further, S has fewer than nine elements if and only if (*) $(x \cap y) \cup (x \cap z) = x$; thus (*) is implied by any equality between two elements of S in the free lattice generated by a,b,c. Unlike the modular and distributive identities, (*) does not imply its dual. *G. Birkhoff* (Cambridge, Mass.).

Kleene, S. C. On the forms of the predicates in the theory of constructive ordinals. Amer. J. Math. **66**, 41–58 (1944). [MF 9938]

This paper deals with the constructive second number class K, a concept due to Church [Bull. Amer. Math. Soc. **44**, 224–232 (1938)]. In a previous paper [J. Symbolic Logic **3**, 150–155 (1938)] the author has defined by transfinite induction a many-to-one mapping of a certain class O of integers on the ordinals of K and has likewise defined a relation of partial ordering $<_0$ between the integers of O, which maps into the usual simple order relation between the ordinals of K. These definitions determine a system of notation S_3 for the ordinals of K.

The author now shows that this transfinite definition of the system S_3 may be replaced by an explicit definition. Using a modification of the technique described in his paper "Recursive predicates and quantifiers" [Trans. Amer. Math. Soc. **53**, 41–73 (1943); these Rev. **4**, 126], he proves that the predicates $a \varepsilon O$ and $a <_0 b$ are expressible in the respective forms $(x)(Ey)R(a, x, y)$ and $(x)(Ey)S(a, b, x, y)$, where R and S are primitive recursive, but are not expressible in the forms dual to these, or in any of the forms with fewer quantifiers. The proof is valid intuitionistically as well as classically. These results are of interest in them-

selves, and they also serve to illustrate a technique of wide applicability for reducing inductive definitions to explicit definitions. *O. Frink* (State College, Pa.).

Gentzen, Gerhard. Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie. Math. Ann. **119**, 140–161 (1943). [MF 10097]

By "reine Zahlentheorie" Gentzen means (roughly) the theory resulting from the functional calculus of first order by the addition of Peano's postulates, postulates for identity and (primitive) recursive definition. Gentzen has previously proved the consistency (likewise the ω-consistency) of Zahlentheorie utilizing transfinite induction up to the first Cantor ϵ-number ϵ_0. [See Math. Ann. **112**, 493–565 (1936) and Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, no. 4, 1938.] From the theorem of Gödel that the consistency of a system (satisfying certain general conditions) cannot be established within the system itself, it follows indirectly that transfinite induction up to ϵ_0 cannot be derived within Zahlentheorie. In the present paper Gentzen gives a direct proof of this together with a proof of the derivability within Zahlentheorie of transfinite induction up to any given ordinal less than ϵ_0. [See Hilbert-Bernays, Grundlagen der Mathematik, vol. 2, Springer, Berlin, 1939, pp. 360 ff.]

The methods Gentzen uses follow closely those of his consistency proof, and for the definitions of many concepts and terms used here reference must be made to the preceding papers. First the concept of a "TJ-Herleitung" is introduced, the formal representation of a purely number-theoretical proof of the validity of transfinite induction up to a given ordinal less than ϵ_0. A method is then given for the step by step construction of TJ-Herleitungen up to ω_{a+1} in terms of TJ-Herleitungen up to ω_a, where $\omega_0=0$, $\omega_a=0$ for $\alpha\geqq\omega$ and $\omega_{a+1}=\omega^{\omega_a}$ for $\alpha<\omega$. Also, if one transgresses the methods of reine Zahlentheorie by admitting quantification over propositional functions, it is shown than transfinite induction up to ϵ_0 is readily derivable.

In the proof that transfinite induction up to ϵ_0 is not derivable it is shown by a reduction process that within a subsystem of Zahlentheorie induction up to some ordinal less than ϵ_0 cannot be established. This reduction process as applied to TJ-Herleitungen is essentially that of the consistency proof, where each deduction can be brought to a normal form and the reduction carried out in a finite number of steps. Associated with each TJ-Herleitung in a certain way is a certain ordinal number, called its "value." It is then shown that the ordinal up to which the induction takes place cannot be greater than its value, and the value is such as always to be less than ϵ_0. *R. M. Martin.*

Wallace, A. D. A substitute for the axiom of choice. Bull. Amer. Math. Soc. **50**, 278 (1944). [MF 10211]

Let Q be any set and R a binary relation over Q. By saying that a subset A of Q is R-simple, the author means that, for every pair a,b of elements of A, we have either aRb or bRa. In terms of this notion, the author formulates the following principle. Any R-simple subset of Q is contained in a maximal R-simple subset of Q. It is shown that this principle is equivalent to the hypothesis that every partially ordered system contains a maximal simply ordered subsystem, and hence to the axiom of choice.

J. C. C. McKinsey (Bozeman, Mont.).

Chwistek, L. B. Sur l'axiome de Zermelo et son rôle dans les mathématiques contemporaines. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 981–985 (1942). (Russian. Georgian and French summaries) [MF 10310]

Bernays, Paul. A system of axiomatic set theory. V. General set theory. J. Symbolic Logic **8**, 89–106 (1943).

This is part V of the Bernays system of set theory [for parts I–IV, see J. Symbolic Logic **2**, 65–77 (1937); **6**, 1–17 (1941); **7**, 65–89, 133–145 (1942); these Rev. **2**, 210; **3**, 290; **4**, 183]. The present installment deals with the arithmetic of the transfinite ordinals, proof by transfinite induction and definition by transfinite recursion.

The author continues with his program of developing general set theory independently of the sum and power axioms (his axioms $V(c)$ and $V(d)$). Without these axioms he is forced to postpone the theory of definition by transfinite recursion until a considerable part of transfinite arithmetic has been developed. The order of topics is: generalized sequences, the "general" recursion theorem, Zorn's lemma and the equivalent maximum principle of Teichmüller, addition of ordinals, proof of the pair class theorem and the pair class axiom, multiplication of ordinals, and then the "restricted" theorem of transfinite recursion, which forms the basis of definition by transfinite recursion. Exponentiation of ordinals is then defined by recursion, and the recursive definitions of addition and multiplication, which have previously been defined directly, are given.

The paper ends with a brief discussion of methods of treating formal algebras within the author's axiom system. By way of example it is shown how the theory of polynomial rings over an abstract ring would be developed.

O. Frink (State College, Pa.).

Gokieli, L. P. On the notion of existence in mathematics. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 23–56 (1942). (Russian. Georgian summary) [MF 10281]

Gokieli, L. P. On the notion of existence in mathematics. I. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 881–888 (1941). (Russian. Georgian summary) [MF 10288]

Gokieli, L. P. On the notion of existence in mathematics. II. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 111–118 (1942). (Russian. Georgian summary) [MF 10309]

Chandrasekharan, K. Intuitionistic theory of linear order. Math. Student **10**, 149–162 (1942). [MF 9980]

This deals with the Brouwerian notions of virtual order and full product. The author proves intuitionistically seven theorems concerning the full product of a finite or denumerable number of virtually ordered sets, generalizing some results of Brouwer concerning the full product of sets of integers. *O. Frink* (State College, Pa.).

Fraenkel, Abraham Adolf. Problems and methods in modern mathematics. Scripta Math. **9**, 162–168 (1943). [MF 10261]

This is a continuation of an article in the same journal **9**, 81–84 (1943); these Rev. **5**, 86.

*Greenwood, Thomas. **Études sur la Connaissance Mathématique.** Éditions de l'Université d'Ottawa, 1942. 112 pp. $1.00.

This book comprises four essays. The first sketches somewhat generally the elementary portions of the first volume of Principia Mathematica, regarding these primarily as extensions of the classical Aristotelian tradition. Two essays are historical studies of the epistemological theories of Aristotle and St. Thomas, emphasizing the aspects in which these theories may be regarded as having made permanent contributions to the philosophy of mathematics. A final essay is devoted to a thomistic interpretation of modern geometry and its relation to the actual world.

R. M. Martin (Princeton, N. J.).

*Greenwood, Thomas. **Essais sur la Pensée Géométrique.** Éditions de l'Université d'Ottawa, 1943. 100 pp. $1.00.

Four essays of the author, recently published in the Revue Trimestrielle Canadienne, are collected in the present booklet. See these Rev. **4**, 50, 127, 210; **5**, 87.

L. M. Blumenthal (Columbia, Mo.).

Bose, S. N. The classical determinism and the quantum theory. Science and Culture **9**, Supplement pp. 1–4 (1944). [MF 10580]

General presidential address at the Indian Science Congress, Delhi.

Whittaker, E. T. Chance, freewill and necessity in the scientific conception of the universe. Proc. Phys. Soc. **55**, 459–471 (1943). [MF 10121]

Oldenburger, Rufus. Pure and applied mathematics. J. Engrg. Education **33**, 432–437 (1943). [MF 9787]

NUMBER THEORY

Whitlock, W. P., Jr. Rational right triangles with equal areas. Scripta Math. **9**, 155–161 (1943). [MF 10260]

Griffiths, L. W. Universal functions of polygonal numbers. II. Amer. J. Math. **66**, 97–100 (1944). [MF 9941]

[The first part appeared in Ann. of Math. (2) **31**, 1–12 (1930).] This paper is a continuation of the author's study of those integer linear combinations

$$(1)\qquad a_1f(x_1)+a_2f(x_2)+\cdots+a_nf(x_n),\quad 1\leqq a_1\leqq\cdots\leqq a_n,$$

of $f(x_k)=x_k+m(x_k^2-x_k)/2$, which represent all integers greater than 0, in which case (1) is said to be universal. The cases in which the weight $w=a_1+\cdots+a_n$ is less than $m+3$ was treated in part I. The present paper deals with $w=m+3$. In this case for (1) to be universal the first four coefficients must be either (1,1,1,1), (1,1,1,2) or (1,1,2,2). In the first two cases certain necessary further conditions on the remaining a's are shown to be sufficient. In the third case the corresponding necessary conditions are perhaps insufficient since a finite number of integers may not be represented by (1). The paper concludes with a discussion of the cases in which the universality of (1) is not a trivial consequence of the universality of another form of lesser weight [see also the same J. **63**, 726–728 (1941); Amer. Math. Monthly **49**, 107–110 (1942); these Rev. **3**, 161, 268].

D. H. Lehmer (Berkeley, Calif.).

Gupta, Hansraj. A formula for $L(n)$. J. Indian Math. Soc. (N.S.) **7**, 68–71 (1943). [MF 10102]

The function L referred to is the sum function

$$(1)\qquad L(x)=\sum_{n\leqq x}\lambda(n)$$

of the well-known purely multiplicative function $\lambda(n)$ of Liouville for which $\lambda(p)=-1$ for p any prime. The interesting function $L(x)$ is, according to a conjecture of Pólya, nonnegative. This conjecture, which implies rather more than the Riemann hypothesis [see A. E. Ingham, Amer. J. Math. **64**, 313–319 (1942); these Rev. **3**, 271], was verified recently for $x\leqq 20000$ by a table of $L(x)$ computed by the author [Proc. Indian Acad. Sci., Sect. A. **12**, 407–409 (1940); these Rev. **2**, 248]. The present paper considers the problem of obtaining isolated values of $L(x)$ for $x>20000$. A complicated formula for $L(n)$ involving $L(n/m)$ summed over odd m's whose prime factors are distinct and bounded, as well as the function $\pi(a, b)$ giving the number of primes between a and b, is derived from Meissel's well-known formula and the identity

$$L(x)+\sum_{r=1}^{\infty}L_{r-1}(x/p_r)=1,$$

where $L_k(x)$ is obtained from (1) by deleting those terms for which n is divisible by one of the first k primes. This formula serves to calculate $L(N)$ given a table of $L(x)$ for $x\leqq N/3$ and a list of primes. It is applied by the author to verify a remark by the reviewer that $L(48512)=-2$. To facilitate the use of his formula the author lists on page 71 all products $m<51051$ of odd distinct primes not greater than 31 arranged according to the parity of the number of prime factors of m.

D. H. Lehmer (Berkeley, Calif.).

Gupta, Hansraj. On the maximum values of $p_k(n)$ and $\pi_k(n)$. J. Indian Math. Soc. (N.S.) **7**, 72–75 (1943). [MF 10103]

This paper is an informal continuation of the discussion in a previous paper by Auluck, Chowla and Gupta [J. Indian Math. Soc. (N.S.) **6**, 105–112 (1942); these Rev. **4**, 211] of the question: for what value (or values) of k is the number $p_k(n)$ of partitions of n into exactly k parts a maximum; in other words, among all partitions of n what is the most popular number of parts? It was conjectured that such a maximum exists for a value of $k=k_0(n)$ and that, as $n\to\infty$,

$$(1)\qquad k_0\sim(\sqrt{6}/2\pi)n^{\frac{1}{2}}\log n.$$

In the present paper the author discusses the prospects of obtaining upper and lower bounds for k_0 by the application of approximate formulas for $p_k(n)$ to the identity

$$p_k(n)-p_{k+1}(n)=p_{k+1}(n+1)-p_{k+1}(n)-p_{k+1}(n-k).$$

However, the general discussion does not take into account the signs of the errors in the two approximate formulas contemplated so that the inequalities obtained are not too reliable. For $n=10{,}000$ the right member of (1) is 359.06, and the author's formulas give $349\leqq k\leqq 353$. [The author gives $349\leqq k\leqq 359$, but this is apparently a miscalculation.] The same argument is applied to the function $\pi_k(n)=p_k(n-k(k-1)/2)$ which enumerates the number of partitions of n into exactly k distinct parts.

D. H. Lehmer.

Linnik, U. V. On Weyl's sums. Rec. Math. [Mat. Sbornik] N.S. 12(54), 28–39 (1943). (English. Russian summary) [MF 8789]

A modification of Vinogradow's method for Weyl sums is expounded to prove that, if $S=\sum_{x=1}^{P}\exp 2\pi i a f(x)$, where $f(x)=a_0x^n+\cdots+a_n$ (the a_i integers), $\alpha=a/q+\theta/q^2$, $(a,q)=1$, $|\theta|<1$, $P\leqq q<P^{n-1}$, then $|S|\ll P^{1-1/(22400n^2\log n)}$. The fundamental lemma, whose proof is given at length, gives an estimate $p^{v-\frac{1}{2}n(n+1)+1/n^{50}}$ to the number of solutions of the system $x_1^i+\cdots+x_v^i=M_i$ $(i=1,\cdots,n)$, where the x_i run over certain products $(<p)$ of primes, and $v=32n[100n\log n]$. *G. Pall* (Montreal, Que.).

Linnik, U. V. An elementary solution of the problem of Waring by Schnirelman's method. Rec. Math. [Mat. Sbornik] N.S. 12(54), 225–230 (1943). (Russian. English summary) [MF 9585]

The paper gives a purely arithmetical proof of the theorem that for every positive integer n there exists a k such that numbers of the form $x_1^n+\cdots+x_k^n$ have positive density, and therefore the sequence of nth powers forms a basis in the sense of Schnirelmann. No estimate is obtained for k, and since more precise results are well known from the work of Hardy and Littlewood, and Vinogradow, interest attaches only to the method. It suffices to prove that for suitable k the number of solutions, for fixed m, of $x_1^n+\cdots+x_k^n=m$ with $|x_i|\leqq P$ is $O(P^{k-n})$. The proof is by induction, with a polynomial $f(x)$ in place of x^n, and the principal weapon is the fact that (briefly) the number of solutions of

$$F(x_1,\cdots,x_s;h_1,\cdots,h_s)=F(x_1',\cdots,x_s';h_1',\cdots,h_s')$$

does not exceed that of

$$F(x_1,\cdots,x_s;h_1,\cdots,h_s)=F(x_1',\cdots,x_s';h_1,\cdots,h_s)$$

multiplied by the number of possibilities for $h_1,\cdots,h_s$. Although the proof is substantially correct, the details in several places are obscure, and there are many misprints. *H. Davenport* (Bangor).

Siegel, Carl Ludwig. Generalization of Waring's problem to algebraic number fields. Amer. J. Math. 66, 122–136 (1944). [MF 9944]

Hardy and Littlewood proved that the number $R_s(\nu)$ of positive integral solutions x_i of

$$\sum_{i=1}^{s}x_i^k=\nu \tag{1}$$

is given by

$$R_s(\nu)=\frac{\Gamma^s(1+1/k)}{\Gamma(s/k)}\mathfrak{S}\nu^{s/k-1}+o(\nu^{s/k-1}), \tag{2}$$

where $s>s_0(k)$ and $\mathfrak{S}$ is the singular series. In this paper their result is generalized to an arbitrary algebraic number field K of degree n. Let $K^{(1)},K^{(2)},\cdots,K^{(n)}$ be the n conjugate fields, where $K^{(l)}$, $l=1,\cdots,n_1$, are real and $K^{(l)}$, $K^{(l+n_2)}$, $l=n_1+1,\cdots,n_1+n_2$, conjugate complex. A number ν of K is called totally positive if $\nu^{(l)}>0$, $l=1,\cdots,n_1$. The number of totally positive integral solutions of (1) in K subject to the further conditions $|x_i^{(l)}|^k<|\nu^{(l)}|$, $i=1,\cdots,s$; $l=n_1+1,\cdots,n_1+n_2$, is denoted by $R_s(\nu)$. Let D be the absolute value of the discriminant of K, and $M=N(\nu)$ the norm of the totally positive integer ν in K.

The generalized form of (2) is as follows. For any fixed $s>(2^{k-1}+n)nk$,

$$R_s(\nu)=D^{(1-s)/2}\sigma_0\mathfrak{S}M^{s/k-1}+o(M^{s/k-1}),$$

where σ_0 is a positive number depending only on n_1, n_2, s, k, and $\sigma_0=\Gamma^{n_2}(1+1/k)/\Gamma^s(s/k)$ when $n_1=0$. The singular series $\mathfrak{S}$ lies between positive bounds when ν belongs to the ring J_k generated by the kth powers of all integers in K, and $\mathfrak{S}=0$ otherwise.

When $k=2$ the generalization was given by the author some time ago [Jber. Deutsch. Math. Verein. 31, 22–26 (1922); Math. Ann. 87, 1–35 (1922); 88, 184–210 (1923)]. When $k>2$ the generalization of the so-called major and minor arcs leads to a difficulty which has now been overcome.

For simplicity the proof is given for the case of a totally real field. The numbers α of K are represented by points $\{\alpha^{(1)},\alpha^{(2)},\cdots,\alpha^{(n)}\}$ of a real n-dimensional Euclidean space R, where $\alpha^{(1)}$ is the conjugate of α in $K^{(1)}$. As usual, $S(\alpha)=\alpha^{(1)}+\cdots+\alpha^{(n)}$, $N(\alpha)=\alpha^{(1)}\cdots\alpha^{(n)}$. Small Gothic letters denote ideals in K; $\mathfrak{d}$ is the ramification ideal of K, $N(\mathfrak{d})=D$. The numbers of $\mathfrak{d}^{-1}$ constitute a lattice (in the older sense) in R and a basis $\rho_1,\rho_2,\cdots,\rho_n$ of $\mathfrak{d}^{-1}$ defines a fundamental parallelepiped E with volume $D^{-\frac{1}{2}}$. Here E takes the place of the interval of integration in the ordinary Waring problem. Let

$$f(\xi)=\sum e\{S(x^k\xi)\},\qquad 0<x^{(l)}<(\nu^{(l)})^{1/k},$$

$$g(\xi)=f^s(\xi)e\{S(-\nu\xi)\},\qquad e(\alpha)=e^{2\pi i\alpha}.$$

Then

$$R_s(\nu)=D^{\frac{1}{2}}\int_R g(\xi)d\xi^{(1)}\cdots d\xi^{(n)}.$$

This device works because for all numbers β in $\mathfrak{d}^{-1}$ the trace $S(\beta)$ is a rational integer.

If γ is any number in K, B_γ is defined as the set of points ξ in R such that

$$N\{\max(h|\xi-\gamma|,t^{-1})\}\leqq N(\mathfrak{a}^{-1}), \tag{3}$$

where $h=M^{(1/n)(1+s-1/k)}$, $t=M^{(1/n)(1/k-a)}$, $a=1/nk-2^{k-1}/(s-1)$, $\mathfrak{a}=(1,\gamma\mathfrak{d})^{-1}$. Here γ takes the place of the rational point a/q, $\mathfrak{a}$ the place of the denominator q and B_γ the place of the major arc. The approximation to $f(\xi)$ on B_γ uses (3) and the Vinogradow device of writing $x=\lambda+\alpha$, where λ runs over a complete residue system mod $\mathfrak{a}$ and $\alpha\equiv 0 \pmod{\mathfrak{a}}$. When ξ is a point not in any B_γ (minor arc), then $f(\xi)=o(M^{1/k-1/s})$. This is proved using lemmas depending on Minkowski's theorem and on the Weyl approximation method. The proof is completed in much the same way as in the case $n=1$.

The author suggests a new problem, that of determining whether the bound for s could be replaced by a function of k alone. This is true when $k=2$. *R. D. James.*

Ko, Chao. On the decomposition of quadratic forms in seven variables. Acad. Sinica Science Record 1, 30–33 (1942). [MF 8829]

Let

$$f_n=\sum_{i=1}^{n}x_i^2+(\sum_i x_i)^2-2x_1x_2-2x_1x_n.$$

It is proved that every classic positive definite $f=\sum a_{ij}x_ix_j$ $(i,j=1,\cdots,7)$, which can not be expressed as $f'(x_1,\cdots,x_7)+f''(x_1,\cdots,x_7)$, where f' and f'' are classic and represent no negative numbers, is equivalent to the form f_7 (of determinant 2). The proof starts with the observation that f must be represented by one of the two forms $y_1^2+\cdots+y_{16}^2$, $y_{16}^2+y_9^2+f_8(y_1,\cdots,y_8)$, and hence (if nondecomposable) by f_8. A detailed analysis of the representation of f by f_8 leads to the stated result. *G. Pall.*

Ko, Chao. On the decomposition of quadratic forms in eight variables. Acad. Sinica Science Record 1, 33–36 (1942). [MF 8830]

It was previously proved [Ko, Sci. Rep. Tsing Hua Univ. 5, nos. 4–6 (1941)] that every positive definite classic f in eight variables not equivalent to f_8 is decomposable [see the preceding review] as $f'+f''$. It is now shown that we can take f' to be a square of a linear form. Use is made of the representation of f by one of the two classes of forms of determinant 1 in eleven variables [Ko, Acta Arith. 3, 79–85 (1938)]. *G. Pall* (Montreal, Que.).

Mordell, L. J. The product of n homogeneous forms. Rec. Math. [Mat. Sbornik] N.S. 12(54), 273–276 (1943). (English. Russian summary) [MF 10222]

Let $X_1, X_2, \cdots, X_n$ be n linear forms in $x_1, x_2, \cdots, x_n$ with real or complex coefficients; X is the matrix of the coefficients of the X_i and is of determinant unity. The contragredient forms are those associated with X'^{-1} as the X_i are with X. Moreover, $K(X)$ is the lower bound of $|X_1X_2\cdots X_n|$ for integer values of the x's not all zero and $k(X)$ is $K(X)$ with X_n replaced by $X_1+X_2+\cdots+X_{n-1}$. Finally, K and k are the upper bounds of $K(X)$ and $k(X)$, respectively, over all sets of forms X_i with $|X|=1$. The author proves the following theorem. If $n>2$ and none of the contragredient forms represents zero, then

$$K(X)\leqq k[K(X'^{-1})]^{1/(n-1)}.$$

Further $K(X)\leqq k^q$ and $K\leqq k^q$, where $q=(n-1)/(n-2)$. *B. W. Jones* (Ithaca, N. Y.).

Hlawka, Edmund. Zur Geometrie der Zahlen. Math. Z. 49, 285–312 (1943). [MF 10038]

We use small gothic letters for points or vectors in n-dimensional Euclidean space R_n ($n\geqq 2$) and gothic capitals for square matrices of order n. The letter $\mathfrak{g}$ denotes a lattice point. All point sets are assumed closed, bounded and Jordan-measurable. If the set S consists of the points $\{\mathfrak{x}\}$, then $S+\mathfrak{a}$ and $\mathfrak{A}S$ denote the sets of points $\{\mathfrak{x}+\mathfrak{a}\}$ and $\{\mathfrak{A}\mathfrak{x}\}$, respectively. The letters k_0 and r are integers satisfying $k_0\geqq 1$, $0\leqq r\leqq k_0-[k_0/2]-1$.

(A) Let K be a set of volume V such that (a) the origin $\mathfrak{o}$ is an inner point of K and (b) every radius vector from $\mathfrak{o}$ meets the boundary of K in just one point. A theorem of Minkowski [Gesammelte Abhandlungen, vol. 1, Teubner, Leipzig, 1911, pp. 265, 269–270, 276] states that, if $V<2\zeta(n)$ (or $V<2\zeta(n)$ if K is symmetrical in $\mathfrak{o}$), then there exists a matrix $\mathfrak{A}$ of determinant 1 such that $\mathfrak{A}K$ contains no lattice point $\mathfrak{g}\neq\mathfrak{o}$. This theorem has an important application to the minimum of definite quadratic forms in n variables [ibid., p. 270] but no proof seems to have been published. The author gives a proof based on the following theorem which is of interest in itself. "Let $\varphi(\mathfrak{x})$ be real, bounded and Riemann-integrable, and let $\varphi(\mathfrak{x})=0$ if $\mathfrak{x}$ lies outside a finite cube. Then to every $\epsilon>0$ there is a unimodular $\mathfrak{A}$ such that

$$\sum_{\mathfrak{g}\neq\mathfrak{o}}\varphi(\mathfrak{A}\mathfrak{g})\leqq\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi(\mathfrak{x})dx_1\cdots dx_n+\epsilon."$$

This inequality implies not only Minkowski's theorem, but also the following one. "If S is of volume less than k_0, then there exists a matrix $\mathfrak{A}$ of determinant 1 such that $\mathfrak{A}S$ contains at most k_0-1 lattice points $\mathfrak{g}\neq\mathfrak{o}$."

(B) Let $\mathfrak{R}$ be a diagonal matrix with the positive numbers $k_1, \cdots, k_n$ in the diagonal. For every set M, denote by $M(\mathfrak{R})$ the set of all points $\mathfrak{R}^{-1}(\mathfrak{x}-\mathfrak{y})$, where $\mathfrak{x}$ and $\mathfrak{y}$ run over M. The author uses methods of C. Siegel [Acta Math. 65, 307–323 (1935)] and J. G. van der Corput [Acta Arith. 1, 62–66 (1935)] to prove the following existence and "alternative" theorems. "If M is of volume $V\geqq k_0k_1\cdots k_n$, then $M(\mathfrak{R})$ contains at least k_0 pairs of lattice points $\mp\mathfrak{g}\neq\mathfrak{o}$." "If M is of volume $V=k_0k_1\cdots k_n$ and if $M(\mathfrak{R})$ contains at most k_0-1 pairs of inner lattice points $\mp\mathfrak{g}\neq\mathfrak{o}$, then every point of R_n lies in at least one and in at most k_0 of the sets $M+\mathfrak{R}\mathfrak{g}$, where $\mathfrak{g}$ runs over all lattice points." "If M is of volume $V\geqq\frac{1}{2}k_0k_1\cdots k_n$ and if at most k_0-r-1 pairs of lattice points $\mp\mathfrak{g}\neq\mathfrak{o}$ are inner points of $M(\mathfrak{R})$, then every point in R_n belongs to at least $r+1$ sets $M(\mathfrak{R})+\mathfrak{g}$, where $\mathfrak{g}$ runs over all lattice points." The last theorem can be further generalized.

(C) Let c and σ be positive numbers, and let $f(\mathfrak{x})$ be a function in R_n with the following properties. (a) $f(\mathfrak{x})>0$ for $\mathfrak{x}\neq\mathfrak{o}$, but $f(\mathfrak{o})=0$. (b) $f(t\mathfrak{x})=|t|f(\mathfrak{x})$ for all real t. (c) If $\mathfrak{x}_1, \cdots, \mathfrak{x}_k$ are any number of points in R_n, and if $1\leqq k_0<k$, then there are k_0+1 of these points, say $\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1, \cdots, \bar{\mathfrak{x}}_{k_0}$, such that

$$(k-k_0)f(\bar{\mathfrak{x}}_i-\bar{\mathfrak{x}}_0)^\sigma\leqq c\sum_{j=1}^{k}f(\mathfrak{x}_j)^\sigma,\quad i=1, 2, \cdots, k_0.$$

[This condition may be replaced by

$$\sum_{i=1}^{k}\sum_{j=1}^{k}f(\mathfrak{x}_i-\mathfrak{x}_j)^\sigma\leqq ck\sum_{j=1}^{k}f(\mathfrak{x}_j)^\sigma$$

for all k]. (d) $f(\mathfrak{x})$ has a positive lower bound on $\mathfrak{x}^2=1$. The author proves the following existence and "alternative" theorems [for the existence theorem, see J. G. van der Corput and G. Schaake, Acta Arith. 2, 152–160 (1936)]. "If $\mathfrak{J}=\int\cdots\int dx_1\cdots dx_n$, the integral extending over $f(\mathfrak{x})\leqq 1$, and $t>((k_0/\sigma\mathfrak{J})(n+\sigma))^{1/n}$, then at least k_0 pairs of lattice points $\mp\mathfrak{g}\neq\mathfrak{o}$ lie in the set $f(\mathfrak{x})\leqq c^{1/\sigma}t$." "If at most k_0-r-1 pairs of lattice points $\mp\mathfrak{g}\neq\mathfrak{o}$ lie in the set

$$f(\mathfrak{x})\leqq c^{1/\sigma}((k_0/2\sigma\mathfrak{J})(n+\sigma))^{1/n},$$

then to every point $\mathfrak{x}_0$ in R_n there are at least $r+1$ lattice points $\mathfrak{g}_i$ such that

$$f(\mathfrak{x}_0-\mathfrak{g}_i)\leqq c^{1/\sigma}((k_0/2\sigma\mathfrak{J})(n+\sigma))^{1/n}."$$

K. Mahler (Manchester).

Stöhr, Alfred. Bemerkungen zur additiven Zahlentheorie. II. Eine Modifikation des Dichtebegriffs. J. Reine Angew. Math. 185, 56–62 (1943). [MF 8816]

[The first part appeared in the same J. 183, 168–174 (1941); these Rev. 4, 241.] Let A, B, C be sets of positive integers and $A+B=C$, $A(x)\geqq\alpha x$, $B(x)\geqq\beta x$, for $x=1, 2, \cdots$. Let $\rho(\alpha, \beta)$ be a function of α and β such that $C(x)\geqq\rho(\alpha, \beta)$. Let $\varphi(x)>0$ and $\varphi(x)/x$ be monotonically decreasing. Furthermore, let $\bar{\alpha}=$g.l.b. $A(m)/\varphi(m)$; $\bar{\beta}$ and $\bar{\gamma}$ are similarly defined. The author proves

$$\bar{\gamma}\geqq\frac{\rho(\bar{\alpha}\varphi(x)/x, \bar{\beta}\varphi(x)/x)}{\varphi(x)/x}. \tag{1}$$

Let $\psi(x)>0$, $\psi(x)/x$ be monotonically decreasing and $\psi(x)=o(x)$. Let $\bar{\alpha}^\sigma=\liminf A(x)/\psi(x)$; $\bar{\beta}^\sigma$ and $\bar{\gamma}^\sigma$ are similarly defined. The author proves that, if $1\in A$ and $1\in B$, then

$$\bar{\gamma}^\sigma\geqq\bar{\alpha}^\sigma+\bar{\beta}^\sigma, \tag{2}$$

where (2) is defined to hold whenever $\bar{\gamma}^\sigma=\infty$. He also shows that (2) is sharp. The first of these two estimates is obtained

by a trivial transformation of the hypothesis. The second estimate is trivial whenever the asymptotic density γ^* of C is positive. For $\gamma^*=0$ it can easily be obtained from a result of the reviewer [Ann. of Math. (2) **43**, 523–527 (1942); these Rev. **4**, 35] or from a previous approximation to it by E. Landau [Nachr. Ges. Wiss. Göttingen Fachgruppe I. **1930**, 255–276 (1930)]. The author proves it in a complicated manner using his first result and Landau's estimate. He fails, however, to give Landau credit for the estimate which he uses. *H. B. Mann* (Barrytown, N. Y.).

Ostmann, Hans-Heinrich. Gegenbeispiel zu einer Frage über Basismengen der additiven Zahlentheorie. J. Reine Angew. Math. **185**, 63–64 (1943). [MF 8817]

Let A be a base set of positive integers of order h, that is to say, hA is the set of all positive integers. Put $nA=A_n$. The author gives an example disproving the conjecture

$$\text{g.l.b.}_{x=1,2,\cdots}\ (1/x)\sum_{n=1}^{h-1}A_n(x)=\sum_{n=1}^{h-1}\ \text{g.l.b.}_{x=1,2,\cdots}\ A_n(x)/x.$$

H. B. Mann (Barrytown, N. Y.).

ANALYSIS

Whitney, Hassler. On the extension of differentiable functions. Bull. Amer. Math. Soc. **50**, 76–81 (1944). [MF 9889]

In this paper previous results of the author [Trans. Amer. Math. Soc. **36**, 63–89 (1934); Ann. of Math. (2) **35**, 482–485 (1934)] are extended to prove the following theorem. Let A be a closed subset of an open set R in n-space E, and having the property that for each $x\varepsilon A$ there is a neighborhood U of x and a number ω such that any two points y and z of $A\cap U$ are joined by an arc in A of length not greater than ωr_{yz}. For m a positive integer, and $\epsilon(x)$ a function that is continuous and positive in R, there is a function $\delta(x)$ which is continuous and positive in A and such that any function $f(x)$ of class C^m in A with derivatives $f_k(x)$ $(\sigma_k=k_1+\cdots+k_n\leqq m)$, satisfying

$$|f_k(x)|<\delta(x), \qquad (x\varepsilon A;\ \sigma_k\leqq m),$$

may be extended to be of class C^m throughout R so that

$$|f_k(x)|<\epsilon(x), \qquad (x\varepsilon R;\sigma_k\leqq m).$$

Former results of the author [Ann. of Math. (2) **37**, 645–680 (1936)] are used to show that this theorem holds if E is replaced by a differentiable manifold, in which a fixed set of coordinate systems (each one intersecting but a finite number of others) is used to measure the size of derivatives.

W. T. Reid (Chicago, Ill.).

Challier, Jean. Extension de la formule de Riemann aux intégrales non linéaires. C. R. Acad. Sci. Paris **214**, 940–942 (1942). [MF 9467]

A generalization of Green's theorem in the plane involving second differentials is derived from the fundamental identity of the calculus of variations. The author states that his generalization has geometrical applications.

P. Franklin (Cambridge, Mass.).

Young, L. C. A further inequality for Stieltjes integrals. J. London Math. Soc. **18**, 78–82 (1943). [MF 9701]

This paper connects with and extends papers of the author [Math. Z. **43**, 255–270 (1937); Math. Ann. **115**, 581–612 (1938)]. If f and g are periodic of period 1, and

$$\left[\int_0^1|f(x+h)-f(x)|^p dx\right]^{1/p}\leqq\varphi_1(h)$$

and

$$\left[\int_0^1|g(x+h)-g(x)|^q dx\right]^{1/q}\leqq\varphi_2(h),$$

where $q=p/(p-1)$ and $\varphi_1(h)$ is monotonic, so that f and g belong to L_p and L_q, then $F(t)=\int_0^1 f(x+t)g(t)dt$ exists. If

$$J(f,g)=\lim_n\frac{F(0)-F(h_n)}{h_n}$$

(which relates to the Stieltjes integral), where $h_n=2^{-n}$, then the inequalities proved are

$$|J(f,g)|<K\int_0^{1/\sqrt{2}}\frac{\varphi_1(\sqrt{u})\varphi_2(u)du}{u^2\log(1/u)}$$

and

$$|J(f,g)-\{F(0)-F(h_n)\}h_n^{-1}|\leqq K\int_0^{h_n}\frac{\varphi_1(\sqrt{u})\varphi_2(u)du}{u^2\log(1/u)}.$$

Similar inequalities are derived for the Moore-Pollard Stieltjes integral $\int_0^1 f dg$ with f and g of bounded Φ and Ψ variations. Application is made to extend the convergence criteria for Fourier series in the Annalen paper, but it is pointed out this is a corollary to the Lebesgue test, using a form due to Gergen [Quart. J. Math., Oxford Ser. **1**, 252–275 (1930)]. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Morse, Marston and Hedlund, Gustav A. Unending chess, symbolic dynamics and a problem in semigroups. Duke Math. J. **11**, 1–7 (1944). [MF 10141]

Suppose e is a configuration of the chessboard and E is a sequence of such configurations beginning with e. In the modified game of chess here contemplated a draw results if the game contains a block EEe. The authors prove that an unending game is possible. The demonstration is based on the properties of the Morse recurrent symbolic trajectory T (with two generators) which is shown not to contain such a block. Another symbolic trajectory is constructed which has three generators and which contains no block EE. This provided an example of a semi-group S with three generators which is not nilpotent ($S^3=z$, where $az=za=z$ for all $a\varepsilon S$) and yet such that $a^2=z$ for all a. *E. R. Lorch.*

Differential Equations

Newell, Homer E., Jr. The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain. Duke Math. J. **10**, 705–709 (1943). [MF 9752]

Ce mémoire fait suite, aussi dans la numération des paragraphes et des théorèmes, à un autre [Duke Math. J. **9**, 245–258 (1942); these Rev. **4**, 99]. En traitant le problème de déterminer des conditions pour qu'une solution du système différentiel représenté par l'équation matricielle

$$(d/dx)Y(x,\lambda)=\{\lambda(\delta_{ij}\sigma_j(x))+(q_{ij}(x,\lambda))\}Y(x,\lambda)$$

puisse se représenter sous la forme

$$Y(x,\lambda)=\left\{I+\sum_{r=1}^{k-1}\lambda^{-r}P^{(r)}(x)+\lambda^{-k}B_k(x,\lambda)\right\}E(x,\lambda),$$

où les matrices $P^{(\nu)}$ et B_k sont analytiques et limitées en x et λ et $E(x, \lambda)=(\delta_{ij}\exp\{\lambda\int_{x_0}^{x} r_j dx\})$, on avait dû poser, dans cette première mémoire, une condition trés restrictive dans le cas que les différences r_i-r_j avaient un pôle simple dans le point x_0. L'auteur a pu maintenant éliminer cette restriction en ampliant légèrement la définition de "régions associées" des valeurs de λ et de x. *B. Levi* (Rosario).

Klotter, Karl und Kotowski, Gertrud. Über die Stabilität der Lösungen Hillscher Differentialgleichungen mit drei unabhängigen Parametern. I. Über die Gleichung $y''+(\lambda+\gamma_1\cos x+\gamma_2\cos 2x)y=0$. Z. Angew. Math. Mech. **23**, 149–155 (1943). [MF 10177]

This is a description of the well-known method for the numerical computation of the characteristic values of λ, that is, the values of λ for which the differential equation

$$y''+(\lambda+\gamma_1\cos x+\gamma_2\cos 2x)y=0$$

has solutions of periods 2π or 4π. Recurrence relations for the coefficients of the Fourier series for such periodic solutions are given, and a table (to four decimals) of the first five characteristic values of λ for $\gamma_1=0.0, 0.5, 1.0, 1.5$; $\gamma_2=0.0, \pm0.1, \pm0.2, \pm0.3, \pm0.4, \pm0.5$. There are photographs of three-dimensional diagrams representing the surfaces in the λ, γ_1, γ_2 space corresponding to periodic solutions and showing the regions in that space of stable and unstable solutions (stability-chart). In future communications the authors intend to discuss similarly

$$y''+(\lambda+\gamma_1\cos x+\gamma_2\sin 2x)y=0$$

and other differential equations of similar structure.
A. Erdélyi (Edinburgh).

Brard, Roger. Mouvements forcès d'un oscillateur soumis à une force dérivant d'un potentiel, à une résistance passive fonction de la vitesse et à une excitation fonction sinusoïdale du temps. C. R. Acad. Sci. Paris **215**, 521–523 (1942). [MF 10186]

The author considers the equation

$$\ddot{x}+2k\dot{x}+a_1x+\lambda(a_2x^2+a_3x^3+\cdots)=C\cos\omega t$$

for small values of λ. The case $a_2=0$ is given special attention. The author obtains well-known results which he appears to regard as new. No references are given to Duffing or to subsequent workers on this equation. The phenomena which the author uses his results to explain and the discovery of which he attributes to Barrillon have been known for decades. *N. Levinson* (Cambridge, Mass.).

Wannier, Gregory H. Energy eigenvalues for the Coulomb potential with cut-off. I. Phys. Rev. (2) **64**, 358–366 (1943). [MF 9708]

The mathematical problem considered in this paper is that of solving the equations

$$\frac{d^2F}{dr^2}+\left[-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r_0}-\frac{l(l+1)}{r^2}\right]F=0, \qquad r<r_0,$$

$$\frac{d^2F}{dr^2}+\left[-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r}-\frac{l(l+1)}{r^2}\right]F=0, \qquad r>r_0,$$

and joining the solutions with continuous derivative at $r=r_0$. The author introduces Bessel-like and Neumann-like functions which reduce to the ordinary Bessel and Neumann functions at zero energy levels (that is, $k=\infty$). A simple approximate formula for the "effective" quantum numbers for the s states is modified to give satisfactory results except when the energy levels are very close to the cut-off in the potential energy. Similar results are obtained for other states of nonvanishing angular momentum. *M. C. Gray.*

Parodi, Hippolyte. Sur une solution de l'équation des télégraphistes. C. R. Acad. Sci. Paris **216**, 606–608 (1943). [MF 9987]

The operational solution of the transmission line equations

$$\frac{\partial E}{\partial x}+rI+l\frac{\partial I}{\partial t}=0, \quad \frac{\partial I}{\partial x}+gE+c\frac{\partial E}{\partial t}=0$$

is obtained in the form of hyperbolic functions of the argument

$$x((lp+1)(cp+g))^{\frac{1}{2}},$$

where p stands for $\partial/\partial t$. The result, which is not new, is obtained by a method equivalent to multiplying the terms not involving $\partial/\partial x$ above by a parameter λ, expanding the solution in powers of λ and equating coefficients of like powers of λ on both sides. The result is applied to exponential and sinusoidal E and I. *H. Poritsky.*

Lapauri, I. D. On the problem of an approximate integration of differential equations of parabolic type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 9–14 (1942). (Georgian. Russian summary) [MF 10312]

Widder, D. V. Positive temperatures on an infinite rod. Trans. Amer. Math. Soc. **55**, 85–95 (1944). [MF 9876]

The temperature $u(x, t)$ at time t and coordinate x on an infinite conducting rod satisfies the equation (1) $u_{xx}=u_t$. The well-known integral representation of this temperature in terms of temperature at time zero, $\phi(y)$, is

$$(2) \qquad u(x, t)=\int_{-\infty}^{\infty}k(y-x, t)\phi(y)dy,$$

with $k(x, t)=(4\pi t)^{-\frac{1}{2}}e^{-x^2/4t}$. The question concerning the uniqueness of the solution has not been solved. For example, $v(x, t)=(x/t)k(x, t)$ is a solution of (1) which, for every x, tends to zero as t tends to zero. By imposing restrictions on $u(x, t)$ with respect to its behavior near $x=\pm\infty$ in the x-t plane, various authors have obtained uniqueness theorems. For one, Tychonoff [Rec. Math. [Mat. Sbornik] **42**, 199–215 (1935)] showed that, provided $u(x, t)e^{-kx^2}$ is bounded for some $k>0$, the solution (2) is the only one.

In the present paper, which is based on the work of Tychonoff and of S. Täcklind [Sur les classes quasi-analytiques des solutions des équations aux dérivées partielles du type parabolique, thesis, Uppsala, 1936], a uniqueness theorem of a different sort is obtained; it is shown that, if the temperature is positive for all positive time, it cannot be identically zero at $t=0$. The first part of the paper is devoted to proving Tychonoff's uniqueness theorem and to setting up the conventional convergence and inversion properties of the Poisson integral (2) in the forms suitable for application to the later work. In the proof of the uniqueness theorem it is first supposed that u satisfies (1) for $t=0$ as well as for $t>0$. The fact that $u\geqq0$ everywhere enters into the uniqueness proof through the fact that it causes $\int_0^t u(x, y)dy$ to be convex in x and non-decreasing in t.

It is shown that any positive $u(x, t)$ satisfying the equa-

tion (1) can be represented as

$$U(x,t)=\int_{-\infty}^{\infty}k(y-x,t)d\alpha(y), \tag{3}$$

where $\alpha(y)$ is nondecreasing, and conversely any such integral is not less than 0 and satisfies (1). Finally, using this representation (3) for u, a more general case of the uniqueness theorem is proved; namely, if (1) $u(x,t)$ satisfies (1) and belongs to class C^2 in $0<t<c$, (2) $u(x,t)\geqq 0$, $0<t<c$, (3) $u(x,0+)=0$, $-\infty<x<+\infty$, then $u(x,t)\equiv 0$, $-\infty<x<+\infty$, $0<t<c$. *J. W. Green.*

Biben, Georges. Compléments à notre note sur l'intégration de l'équation de M. De Donder. C. R. Acad. Sci. Paris **213**, 646–648 (1941). [MF 9648]

In this note the author discusses some special cases of the equations obtained in the note mentioned in the title [same C. R. **209**, 726–728 (1939)], cases in which by suitable selection of constants especially simple results can be obtained. For example, a special case of the equation

$$\left(1-\frac{a}{r}\right)\frac{d^2\varphi}{dr^2}+\left(2-\frac{a}{r}\right)\frac{1}{r}\frac{d\varphi}{dr}+k^2\varphi=0$$

is obtained by putting $a=1$, $k=0$; the author discusses the significance with respect to the problem at hand of the simple solution obtained. *J. W. Green.*

Biben, Georges. Sur une extension de la méthode des spectres à la mécanique ondulatoire relativiste de M. De Donder. C. R. Acad. Sci. Paris **213**, 773–775 (1941). [MF 9655]

In this note the author considers complex functions φ and their conjugates φ^* in three dimensional Riemannian space. If Δ_2 is the second Beltrami operator, then the following theorem is proved. A necessary and sufficient condition that there exist a function φ with conjugate φ^* vanishing on the frontier of a domain D and satisfying in D the equations $\Delta_2\varphi+k^2\varphi=0$, $\Delta_2\varphi^*+k^2\varphi^*=0$ is that k^2 be a stationary value of the quotient

$$f(\varphi,\varphi^*)=\frac{\int_D\Delta_1(\varphi,\varphi^*)dT}{\int_D\varphi\varphi^*dT}.$$

The methods used in the proof consist mainly of formal transformations of the integrals by Green's formulas.
J. W. Green (Aberdeen, Md.).

Mechvarišvili, Ja. G. On Fourier's method. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 485–490 (1941). (Russian. Georgian summary) [MF 10294]

A formal solution of the equation $\Delta u+\lambda^2 u=0$, where Δ denotes the Laplacian, can be obtained in terms of trigonometric and Bessel functions as a sum of solutions of the type $f(r)g(\theta)$, where (r,θ) are polar coordinates. The author shows that for the ring between two concentric circles all solutions are in fact of this form. He uses a general representation for solutions given by Vecoua [Mitt. Georg. Abt. Akad. Wiss. USSR **1**, 329–335 (1940); Trav. Inst. Math. Tbilissi **7**, 161–253 (1940); these Rev. **3**, 127]. For the circular ring, the arbitrary analytic functions involved in the solution can be expanded into Laurent series, and the resulting series for the solution coincides with the formal solution. *R. P. Boas, Jr.* (Cambridge, Mass.).

Mitra, Samarendra Kumar. On the boundary value problems of Laplace's equation relating to two spheres. A new method of solution. Science and Culture **9**, 397–398 (1944). [MF 10278]

This paper outlines formally a solution of the problem of determining the potential due to two charged spheres.
F. W. Perkins (Hanover, N. H.).

Vekua, N. P. On the solution of the mixed boundary value problem in the theory of the Newton potential for a multiply connected domain. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 753–758 (1942). (Russian. Georgian summary) [MF 10319]

This paper is based largely on results of Muschelishvili. In a recent paper Muschelishvili solved the mixed boundary value problem in the theory of logarithmic potential for multiply connected domains. Here the three dimensional case is considered. The author uses the artifice introduced by Muschelishvili [J. Appl. Math. Mech. (N.S.) **4**, no. 4, 3–26 (1940); these Rev. **3**, 151] to modify the integral equations for the single- and double-layer density functions so that the homogeneous form of the equations has no solutions. Reference is also made to a result of Keldych and Lavrentiev. *N. Levinson* (Cambridge, Mass.).

Nielsen, Kaj L. Some properties of functions satisfying partial differential equations of elliptic type. Duke Math. J. **11**, 121–137 (1944). [MF 10153]

Let $z=x+iy$, $z^*=x-iy$ and consider the Bergman operator

$$P(f)=\int_{-1}^{1}E(z,z^*,t)f(z[1-t^2]^{\frac{1}{2}})[1-t^2]^{-\frac{1}{2}}dt,$$

where E has the form $E(z,z^*,t)=\exp\left[\sum_{k=1}^{\infty}C_k(z,z^*)t^k\right]$. For the set of functions $f(z)=z^n$, it is shown that $P(z^n)$, considered as a function of z (or of z^*) satisfies an ordinary linear differential equation of order $n+1$ or less. When $f(z)=(z-a)^{-1}$, $P(f)$ satisfies an ordinary linear differential equation as a function of z^* and a system of ordinary equations as a function of z. (Similar results hold for $f(z)=(z-a)^{-n}$.) If $f(z)$ and $P(f)$ have the Taylor developments $f(z)=\sum a_iz^i$, $P(f)=\sum\alpha_{ij}z^iz^{*j}$, then a_i is given explicitly in terms of $\alpha_{00}, \alpha_{10}, \cdots, \alpha_{i0}$. Finally some remarks on the distributions of the singularities of $P(f)$ are made when $f(z)$ has a finite number of poles. [In equation (3.1) $D_0=D$, and $U_n^{[m]}$ should be $W^{[m]}$; in (3.2) replace μ by m; in (4.12) the formula for $\theta_n^{[1]}$ needs some adjustments; in (5.6) assume $C_i(0,0)=0$, etc.] *F. G. Dressel.*

Nielsen, Kaj L. On the Bergman operators for linear partial differential equations. Bull. Amer. Math. Soc. **50**, 195–201 (1944). [MF 10197]

If $E(z,\bar{z},t)$ is a solution of the equation

$$(1-t^2)E_{z\bar{z}}-E_{\bar{z}}/t+2zt(E_{z\bar{z}}+BE_{\bar{z}}+CE)=0, \tag{1}$$

then the Bergman operator

$$V=P(f)=\int_{-1}^{1}E(z,\bar{z},t)f(z[1-t^2]^{\frac{1}{2}})[1-t^2]^{-\frac{1}{2}}dt$$

furnishes a solution of the differential equation

$$V_{z\bar{z}}+BV_{\bar{z}}+CV=0 \tag{2}$$

for each analytic function $f(z)$. The author considers E in the forms

$$E(z, \bar{z}, t) = \exp(R(z, \bar{z})t^n), \tag{3}$$

$$E(z, \bar{z}, t) = \exp(N(z, \bar{z}) \cdot t^n + M(z, \bar{z})t^m) \tag{4}$$

and determines B and C for (2) by substituting E in (1). Define E as trivial if $R_{\bar{z}} = N_{\bar{z}} = M_{\bar{z}} = 0$. When E has form (3) the only nontrivial cases occur for $n = 1$ or $n = 2$. For E of form (4) nontrivial cases can occur only for the following values of m and n: (i) $m = 0$, $n = 1$ or 2, (ii) $m = 1$, $n = 2$ or 3, (iii) $m = 2$, $n = 4$. *F. G. Dressel* (Durham, N. C.).

Marden, Morris. A recurrence formula for the solutions of certain linear partial differential equations. Bull. Amer. Math. Soc. **50**, 208–217 (1944). [MF 10199]

Using the notation of the preceding review, the solutions $V_n = P(Z^n)$, $n = 1, 2, \cdots$, of equations (2) are called "basic solutions." For the special case $\Delta V + V = 0$ of (2) the function E takes the form $E(z, \bar{z}, t) = e^{itr}$ $(z = re^{i\theta}, r = (z\bar{z})^{\frac{1}{2}})$. In this case V_n can be written in terms of the nth Bessel function

$$V_n = (\pi^{\frac{1}{2}}/2)\Gamma(n + \tfrac{1}{2})e^{n\theta i} J_n(r).$$

Due to a well-known relation among Bessel functions, the "basic solutions" V_n satisfy the recurrence formula:

$$\partial V_n/\partial r = (n/r)V_n - (2e^{-i\theta}/(2n+1))V_{n+1}.$$

The present paper finds the recurrence formulas for the "basic solutions" when

$$E(z, \bar{z}, t) = t^q\left(\sum_{i=0}^{n} b_i(r, \theta)t^{2i}\right) \exp\left(t^r \sum_{j=0}^{m} a_j(r, \theta)t^{2j}\right).$$

Here (q, r) can have any one of the pair of values $(0, 0)$, $(0, 1)$, $(1, 1)$, and the b_i, a_j have continuous derivatives with respect to r and θ. *F. G. Dressel* (Durham, N. C.).

Soboleff, S. Sur la stabilité en moyenne des solutions du problème limite de l'équation du type hyperbolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 383–385 (1941). [MF 9599]

Let

$$L(u) = \sum_{i,j=1}^{n} (\partial/\partial x^i)A_{ij}(\partial/\partial x^j)u + Cu = \rho(\partial^2 u/\partial t^2 + F), \tag{1}$$

where the coefficients are independent of t, $\sum\sum A_{ij}p_ip_j$ is positive definite and A_{ij}, C are of class C' in the domain Ω spanned by a regular surface S. Consider the mixed problem for (1): (2) $u|_{t=0} = u_0$, $u_t|_{t=0} = u_1$ and either (3a) $u|_S = 0$ or (3b) $\partial u/\partial n|_S = 0$. Let λ_m be the first positive characteristic number of $L(u) + \lambda\rho u = 0$. Write U_i for the characteristic function corresponding to λ_i. Then

$$u = \sum_{1}^{m-1} a_i(t)U_i + u_m,$$

where

$$d^2a_i/dt^2 + \lambda_i a_i = \int \cdots \int \rho U_i F d\Omega = \phi_i(t).$$

Write

$$f_1(t) = \int \cdots \int \rho\left[F - \sum_{1}^{n-1} U_i\phi_i\right]^2 d\Omega.$$

Upper bounds of the form $A + B\int_0^t f_1(t)dt$ are derived for the mean square values of u_m, $\partial u_m/\partial t$ and ∇u_m. The central theorem asserts that $\int_0^\infty f_1(t)dt < \infty$ and $m = 1$ imply the solution and its first derivatives belong to L_2 on $0 \leqq t < \infty$. *D. G. Bourgin* (Urbana, Ill.).

Soboleff, S. Sur le problème de la stabilité des solutions du problème limite pour les équations aux dérivées partielles du type hyperbolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 459–462 (1941). [MF 9603]

Some complicated inequalities are developed involving derivatives of solutions of equations of the type (1) [cf. the preceding review] subject to (3a) or (3b). *D. G. Bourgin* (Urbana, Ill.).

Soboleff, S. Quelques problèmes nouveaux pour les équations aux dérivées partielles. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 463–466 (1941). [MF 9604]

Consider (1) [cf. the second review above] with $A_{ij} \varepsilon N(B, \tau_1, P)$, $C \varepsilon N(B, \tau_1 + 2, P - 1)$, $\rho \varepsilon N(B, \tau_1 + 2, P - 1)$ in the notation of the following review. Write

$$\|F\|_{k,\beta} = \left[\int \cdots \int \sum \frac{(\beta-1)!}{\alpha_1! \cdots \alpha_n!}\left(\frac{\partial^{k-\beta} F_{\alpha_1 \cdots \alpha_n}}{\partial t^{k-\beta}}\right)^2 \rho d\Omega\right]^{\frac{1}{2}}.$$

The main theorem asserts that, if $\|F\|_{k,\beta} < Me^{\sigma t}$, $k \leqq P$, $\beta \leqq k$, $\sigma > 0$, and if the solution of (1) is subject to (3a) or (3b) [cf. the second review above] and also to $\|u\|_{k,\beta} \leqq Ne^{\sigma_1 t}$, $\sigma_1 < \sigma$, $-\infty < t < T$, then these conditions can be satisfied and the solution involves h arbitrary constants, where h is the number of negative characteristic values of $L(u) + \lambda\rho u = 0$ not greater than $-\sigma_1^2$. *D. G. Bourgin* (Urbana, Ill.).

Soboleff, S. Sur quelques groupes de transformations de l'espace n-dimensionnel. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 380–382 (1941). [MF 9598]

[Cf. the three preceding reviews.] Let

$$\phi_{\alpha_1 \cdots \alpha_n} = \partial^k \phi / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}.$$

Let $N(A, \tau, l)$ be the class of functions defined on the bounded domain Ω and subject to

$$\left\{\int \cdots \int |\phi_{\alpha_1 \cdots \alpha_n}|^{n/(k+\tau)}\right\}^{(k+\tau)/n} < A < \infty,$$

where either $k < l$ or $\tau > 0$ and $0 < k + \tau \leqq n$, $\alpha_1, \cdots, \alpha_n$ arbitrary. Some straightforward results are given for relations between $N(A, \tau, l)$ and $N(A', \tau', l')$ under various conditions on the primed and unprimed quantities. Let

$$N(\tau, l) = \cup_A N(A, \tau, l).$$

The transformations $y^i = y^i(x_1, \cdots, x_n)$, subject to $|\partial y^i/\partial x^j| < \infty$,

$$\begin{pmatrix} y^1 & \cdots & y^n \\ x_1 & \cdots & x_n \end{pmatrix} > 0$$

and $\int \cdots \int |y^i_{\alpha_1 \cdots \alpha_n}|^{l-1} < \infty$ form a group G. The main theorem is that $N(\tau, l)$ is invariant under G. The author conjectures that G is the maximal group leaving $N(\tau, l)$ invariant. *D. G. Bourgin* (Urbana, Ill.).

Krüger, M. Die Theorie der in endlicher Entfernung von der Trennungsebene zweier Medien erregten Kugelwelle für endliche Brechungsindizes. Z. Phys. **121**, 377–437 (1943). [MF 9799]

In medium 1, filling the half space $z > 0$, there is a source of spherical waves at the point $(0, 0, d)$ and there is no conductivity ($\sigma_1 = 0$). Medium 2, filling the half space $z < 0$, is generally conductive. The Hertzian vector for a vertical dipole has a single component of type

$$(i/kR)\exp(2\pi i\nu t - ikR),$$

where R is the distance from the source. With the aid of Weyl's integral for this expression the incident waves may

be regarded as being resolved into homogeneous plane waves traveling towards the plane $z=0$ and a set of mock Rayleigh waves with amplitude increasing upwards instead of downwards. These latter waves travel with speeds less than that of the plane homogeneous waves. This representation is valid only for $z<d$ and must be replaced by another one for $z>d$. Each of the homogeneous plane waves is supposed to give rise to a reflected homogeneous plane wave and a transmitted homogeneous plane wave, the directions and amplitudes being calculated by Fresnel's laws. The mock Rayleigh waves are treated by an extension of Fresnel's laws with the result that each mock Rayleigh wave is supposed to give rise to a true Rayleigh wave in medium 1 and a true Rayleigh wave (or rather wave of Somigliana when there is conductivity) in medium 2. The mock Rayleigh waves must be regarded as having a mathematical rather than a physical significance. Though the mathematical requirements for medium 1 are violated in the case of each individual mock Rayleigh wave they are satisfied on the whole when all types of waves are taken into consideration and the whole incident field is continued analytically into the region $z>d$. The true Rayleigh waves and the waves of Somigliana satisfy the requirements individually.

To obtain a physical interpretation of the results obtained by this mathematical procedure the various double integrals have to be evaluated approximately and this is done by various deformations of paths of integration in complex planes. Numerous diagrams are given to make the transformations clear. It is thought that, by taking into consideration a singularity neglected by Weyl, a satisfactory explanation can be found for the "head wave" found experimentally by O. v. Schmidt. This has a conical wave-front whose base is the trace of a spherical wave-front traveling in medium 2 with the speed of propagation of this medium. The cone is also tangential to the reflected spherical wave in medium 1. If medium 1 is continued in imagination into the half space $z<0$ the conical wave front can be regarded as coming from a cone whose vertex is at the image of the source in the plane $z=0$.

The problem is formulated in section 1. Fresnel's laws of reflexion are given in section 2, accompanied by analysis depending on the use of a Hertzian function. Section 3 deals with the evaluation of the Hertzian function for medium 1 in the general case when the index of refraction n is a complex quantity with a real part exceeding 1. For convenience all distances are measured in wave-length units so that k may be taken equal to 1. In section 4 the Hertzian function for medium 2 is evaluated. The orders of magnitude of the different terms retained and those neglected are indicated for both media so that it can be seen which terms will predominate at a great distance from the source.

H. Bateman (Pasadena, Calif.).

Brown, Herbert Kapfel. General vibration of a hinged beam. J. Appl. Phys. **15**, 410–413 (1944).

The author applies the finite Fourier transformation to the solution of the linear partial differential equation

$$\frac{\partial^2 V}{\partial t^2}+C_1(t)\frac{\partial^4 V}{\partial x^4}+C_2(t)V=P(x,t), \quad 0<x<\pi; \quad t>0.$$

The boundary conditions are $V(+0,t)=C_3(t)$, $V(\pi-0,t)=C_4(t)$, $V_{xx}(+0,t)=C_5(t)$, $V_{xx}(\pi-0,t)=C_6(t)$, $t>0$ and the C_i's, $i=1,\cdots,6$, are prescribed functions of t, while the initial conditions are $V(x,+0)=F_1(x)$ and $V_t(x,+0)=F_2(x)$, $0<x<\pi$, and the F's are prescribed functions of x.

A. E. Heins (Cambridge, Mass.).

Theory of Probability

Ville, Jean. Sur un critère d'indépendance. C. R. Acad. Sci. Paris **216**, 552–553 (1943). [MF 10027]

The problem of this note is to obtain criteria testing whether a given sequence of n numbers should be regarded as a sequence of n independent trials of some random variable. The given sequence is first transformed into a 1,0 sequence by assigning to each term the value 1 or 0 according as that term does or does not exceed an appropriately chosen number m (usually taken as the median of the sequence). The author denotes by r_1, r_2, r_3, r_4 the respective frequencies with which the configurations 00, 01, 10, 11 occur in the 1,0 sequence. He studies the joint distribution of the variables r_1, r_2, r_3, r_4 first under the assumption that the 1,0 sequence is a simple Markoff chain and second for the special case in which independence is assumed. The desired criteria are defined in terms of critical regions related to the latter distribution.

A. H. Copeland.

Santaló, L. A. On the probable distribution of corpuscles in a body, derived from their distribution in its cross-sections, and similar problems. Revista Union Mat. Argentina **9**, 145–164 (1943). (Spanish) [MF 10124]

Let V and F be the volume and area, respectively, of a convex body K in Euclidean n-space. Let G be a variable straight line intersecting K. Then the average length of the segment $G\cdot K$ is equal to $\delta=\omega_n V/\kappa_{n-1}F$, where $\omega_n(\kappa_{n-1})$ is the area (volume) of the n $(n-1)$-dimensional unit sphere. Now let K contain N small congruent convex bodies k, each of area f. Then the average number of k's intersected by a G will be $\bar{n}=Nf/F$. Thus the density of the k's on the G's will be $\delta=\bar{n}/\delta=\kappa_{n-1}fD/\omega_n$, where $D=N/V$ is the density of the k's in K; in particular, $\delta=\kappa_{n-1}r^{n-1}D$ if the k's are n-dimensional spheres of radius r. The reciprocal δ^{-1} is interpreted as the average range of visibility at a point in K. This is a generalization of a result by Pólya [Arch. Math. Phys. **27**, 135–142 (1918)]. The author also solves similar problems in two and three dimensions, replacing the G's by variable planes or strips of constant width or convex cylinders, and discusses a related problem of the kinetic theory of gases. His methods are those of integral geometry.

P. Scherk (Saskatoon, Sask.).

Orts Aracil, J. M.* On the behavior of certain probabilities. Revista Mat. Hisp.-Amer. (4) **3**, 157–163 (1943). (Spanish) [MF 10136]

Let $P_k=\binom{n}{k}p^k(1-p)^{n-k}$. The author studies the asymptotic behavior, for $n\to\infty$, of the maximum term and of the symmetric functions of the P_k. The point is to avoid Stirling's formula.

W. Feller (Providence, R. I.).

Jánossy, L. Rate of n-fold accidental coincidences. Nature **153**, 165 (1944). [MF 10059]

The author is concerned with accidental coincidences for n Rutherford-Geiger counters; a pulse at time 0 at any counter is recorded as an n-fold coincidence if in the interval $(-t, t)$ each of the $n-1$ remaining counters gives rise to at least one pulse. Here t is a constant, the resolving time.

W. Feller (Providence, R. I.).

Loève, Michel. La loi forte des grands nombres pour des variables aléatoires liées. C. R. Acad. Sci. Paris 212, 1121–1123 (1941). [MF 9187]

Loève, Michel. La tendance centrale des sommes de variables aléatoires liées. C. R. Acad. Sci. Paris 213, 9–11 (1941). [MF 9149]

These two notes give generalizations of the strong law of large numbers and the central limit theorem to sequences of dependent random variables. The notation is that of an earlier note of the author [same C.R. 212, 840–843 (1941); these Rev. 3, 3]. No proofs are given. *M. Kac.*

Kosambi, D. D. Statistics in function space. J. Indian Math. Soc. (N.S.) 7, 76–88 (1943). [MF 10104]

The author discusses statistical problems connected with continuous stochastic processes whose representative functions $x(t)$ are defined by $x(t)=\sum x_j\varphi_j(t)$, where the φ_j determine an orthonormal set and $x_1, x_2, \cdots$ are mutually independent Gaussian chance variables with vanishing means and variances $\sigma_1^2, \sigma_2^2, \cdots$, respectively. It is supposed that $\sum\sigma_j^4<\infty$ and that $K(s,t)=\sum\sigma_j^2\varphi_j(s)\varphi_j(t)$ defines a continuous integral operator. The process is determined completely by the function K. The samples he considers are functions $x(t)$ rather than merely the values of functions $x(t)$ at a finite number of points. An estimate of the function K is given in terms of a sample of n functions $x(t)$. Various mechanical and electrical methods are suggested for combining functions $x(t)$, given graphically, as necessitated by this type of statistical approach. *J. L. Doob.*

Dieulefait, C. E. On Slutsky's sinusoidal limit law, derived from a new sequence of random variables. An. Soc. Ci. Argentina 134, 257–285 (1942). (Spanish) [MF 8431]

Let $\{x_j\}$, $j=1, 2, \cdots$, denote a sequence of stochastic variables subject to the following conditions: $E(x_j)=0$ (E is mathematical expectation); $E(x_j^2)=A^2e^{2\alpha j}$; $E(x_jx_{j+t})=A^2e^{\alpha(2j+t)}f(t)$, $f(t)=f(-t)$, with A, α real positive numbers. Consider the sequence

$$x_j^{(n)}=x_j^{(n-1)}+x_{j-1}^{(n-1)}+\cdots+x_{j-(s-1)}^{(n-1)},$$
$$n=1, 2, \cdots;\quad s\geqq 2;\quad x_j^{(0)}\equiv x_j.$$

The main point is to prove that, under certain conditions concerning the coefficients of autocorrelation for n, $m\to\infty$, $m/n=$const., $\{\Delta^m x_j^{(n)}\}$ tends asymptotically, with probability as close to unity as we please, to a sinusoidal curve of a determined period. This is a generalization of the "law of the limiting sinusoid" first established by Slutsky ($\alpha=0$; $f(t)=0$ for $t\neq 0$). The analysis, rather lengthy, follows the lines of reasoning of V. Romanowsky [Rend. Mat. Circ. Palermo 57, 82–111, 130–136 (1933)] who previously generalized Slutsky's results. It is based upon the representation

$$x_j^{(n)}=\sum_{r=0}^{n(s-1)}R_r^{(n)}x_{j-r},$$

where

$$(1+x+\cdots+x^{s-1})^n=(1-x^s)^n(1-x)^{-n}=\sum_{r=0}^{n(s-1)}R_r^{(n)}x^r,$$

and upon the study of the $R_r^{(n)}$ by means of contour integrals in the complex domain. *J. A. Shohat.*

Mathematical Statistics

***Fisher, Ronald A. and Yates, Frank. Statistical Tables for Biological, Agricultural and Medical Research.** 2nd ed. Oliver and Boyd Ltd., London, 1943. viii+98 pp. 13 s. 6 d.

These tables may be arranged in five groups: (i) standard tables used in applications of the theory of regression and more particularly of the analysis of variance; (ii) tables for the statistical analysis of data that are expressed as fractions or percentages; (iii) tables which enable the reader to construct the plans for certain types of experimental arrangements; (iv) a few tables for the solution of special problems; (v) tables of some of the common mathematical functions which are used in statistical work.

Regression and analysis of variance. Tables I–VI give the percentage points and the ordinates of the normal distribution and a number of percentage points of the t, χ^2, z and r distributions. These tables are reprinted, with a few additional values, from Fisher's Statistical Methods for Research Workers. Table VII, from the same source, gives the transformation $r=\tanh z$, which may be used for comparisons between a number of independent correlation coefficients. The z distribution is supplemented by a table of the variance ratio e^{2z}, both tables appearing at the 20, 5, 1 and 0.1 percent points. Two additional tables are probably less familiar. For $n=1(1)52$, table XXIII provides the ordinates of the orthogonal polynomials of degree 1, 2, ···, 5 fitted to a series of n points at equal intervals. This table is intended to shorten the computations required in fitting by least squares a polynomial to a series of observations at equal intervals, the calculations involving little more than the sums of products of the observations with the appropriate polynomial ordinates. An interesting extension of the t-test is provided in table Vi (due to P. V. Sukhatme) which gives the 5 and 1 percent significance levels of the Behrens-Fisher test for the difference between two independent sample means from normal populations whose variances are not assumed to be equal. A more extensive table is also included for the case where one population variance is not known.

Fractions or percentages. The tables in this section apply when the data are subject to binomial variation. Table VIIIi (due to W. L. Stevens) supplies the upper and lower fiducial or confidence limits of the binomial and Poisson distributions corresponding to probabilities of 90, 97.5 and 99.5 percent. Table VIII is a reprint of Yates's table [Suppl. J. Roy. Statist. Soc. 1, 217–235 (1934)] of the exact significance levels of the χ^2 test for independence in a 2×2 contingency table. There are two tables (XII and XIII) of the transformation $y=\sin^{-1}\sqrt{p}$, where p is expressed in percentage and y in angles. This transformation is often applied to fractions which are to be subjected to an analysis of variance, since the variances tend to be more uniform in the new scale.

Experimental arrangements. For Latin squares of size 4×4 to 6×6, sets of standard squares are given from which all possible squares can be generated by permutations of rows, columns and letters. Squares from the 7×7 to the 12×12 are represented by examples. Complete sets of orthogonal Latin squares, which form the basis of many devices used in the layout of replicated experiments, appear for squares up to the 9×9, excepting the 6×6 in which case no complete set exists. Tables XVII–XIX describe a useful series of experimental plans known as balanced incomplete blocks. These plans are arrangements of v letters in groups

of k ($v>k$), subject to the restriction that any pair of letters appears equally often within the same group. In replicated experiments, where the letters represent the experimental treatments, the most practicable solutions are those in which no letter appears more than 10 times; all known solutions of this type are given in condensed notation.

Special problems. For $n=2(1)50$, $r=1(1)\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, the mean value of the rth largest of a sample of size n from the standard normal distribution is given (2D) in Table XX. This table is proposed as a means of transforming ranked data to a scale on which analysis of variance techniques may be applied and is also useful in many problems dealing with ordered data. The remainder of this miscellaneous group consists of three tables of the well-known probit transformation for toxicological tests, a table for the estimation of the number of organisms present in a medium by the dilution method of counting and a table of $\Delta^r 0^s/r!$ for $s=2(1)25$.

Common mathematical functions. These are alike for the statistician and the mathematician: logs, lns., squares, square roots, reciprocals, factorials, sines and tangents. Finally there is a collection of 15,000 random digits.

W. G. Cochran (Princeton, N. J.).

***Deming, W. Edwards. Statistical Adjustment of Data.** Wiley & Sons, Inc., N. Y., 1943. x+261 pp. $3.50.

This book is intended to be a text and a book of reference, to deal with the scattered and less generally known parts of least squares rather than the well-known parts and to supplement, rather than supplant, other books. By intention, some of the best known parts of least squares are omitted, and such familiar problems as the fitting of curves to data in which the X's are accurate and the Y's alone subject to error are slighted. The book is not introductory; a prior knowledge is presumed, for example, of the meanings of certain statistical terms like "variance" and "correlation."

There are eleven chapters. The first three lead up to chapter IV, in which the general problem of least squares is stated. There are n observed quantities X_i with weights w_{xi} and n observed quantities Y_i with weights w_{yi}. There are also some unknown parameters a, b, c, etc. The problem is to find values of a, b, c, etc., and "adjusted" values x_i of the X's and y_i of the Y's such as to minimize the sum

$$\sum_i [w_{xi}(X_i-x_i)^2+w_{yi}(Y_i-y_i)^2]$$

and at the same time satisfy j conditions of the type

$$F^h(x_1, x_2, \cdots, y_n, a, b, c, \cdots)=0, \quad h=1, 2, \cdots, j.$$

The solution is obtained by the use of Lagrange multipliers.

In later chapters a detailed discussion is given of less general situations which occur in geometrical problems (surveying) where there are no parameters and in problems of curve fitting where interest centers in the parameters rather than in the adjusted values. Rules are given for finding the weights, in the geometrical problems, of functions of the adjusted values and, in curve fitting, of functions of the parameters. A chapter is devoted to the adjustment of sample frequencies to expected marginal totals. Methods of computation are illustrated by examples worked out numerically, as well as by computing schedules with symbolic entries. Numerous exercises are provided for the student.

A fault of the book is that it does not presume a uniform degree of prior knowledge and hence its value as a text is limited. A greater fault is the disordered manner in which some topics are presented. On pages 65 and 66, for instance, a schedule is displayed and discussed for computing the weights of functions of the adjusted angles of a triangle, although the formulae for such weights, without which the computations cannot be understood, have not been given. More complicated examples are worked out later, still without the necessary formulae, until at last, on page 90, Gauss' expressions for the weights are revealed to the reader.

The book as a whole is free from errors, but there are several on page 56, where L_{hh} is said to be the weight of F^h and L_{hk} to be the reciprocal of the covariance of F^h and F^k. If parameters are present, they contribute to the variances and covariances of the F's without appearing in the L's and hence the weights and covariances of the F's involve in general the partial derivatives F_a, F_b, etc., and thus are not functions of the L's alone. If parameters are absent, then L_{hh} is the reciprocal of the weight of F^h, and L_{hk} is the covariance of F^h and F^k.

Despite the faults of the book, the computational procedures that it describes are excellent and chapter IV, its heart, is a connected and concise exposition of the general problem of least squares. Although the reviewer does not recommend reading through the book consecutively, it does contain information, not readily available elsewhere, that is of much value to those who deal with observational data.

T. E. Sterne (Aberdeen, Md.).

Sen Gupta, J. M. A note on adjustments for first and second moments in a grouped frequency distribution split up into sub-sections. Sankhyā 6, 413–414 (1944). [MF 10622]

Lord, Frederic M. Alignment chart for calculating the fourfold point correlation coefficient. Psychometrika 9, 41–42 (1944). [MF 10234]

Anderson, T. W. On card matching. Ann. Math. Statistics 14, 426–435 (1943). [MF 9847]

The author obtains asymptotic distributions for certain card matchings. These distributions can be used to compute approximate significance levels and the results are applicable to problems in psychology. First he considers the suit matching which is produced by the random pairing of the cards of two decks of arbitrary composition. He shows that the distribution is asymptotically normal provided the proportion of cards in each suit remains fixed as the number of cards becomes infinite. The result is extended to the case of three or more decks. Next he considers the matching of two identical decks each of n different cards. The resulting distribution is shown to be asymptotically Poisson.

A. H. Copeland (Ann Arbor, Mich.).

Levene, H. and Wolfowitz, J. The covariance matrix of runs up and down. Ann. Math. Statistics 15, 58–69 (1944). [MF 10238]

Runs up and down are used in quality control and in economic time series. Such runs are defined for a sequence of unequal numbers $h_1, h_2, \cdots, h_n$ in terms of the signs of the numbers in the auxiliary sequence $h_{i+1}-h_i$, where $i=1, 2, \cdots, n-1$. If within the auxiliary sequence there is a subsequence consisting of p consecutive numbers neither immediately preceded nor immediately followed by a positive number (that is, a subsequence of maximum length), then such a subsequence constitutes a run up of length p.

A run down of length p is similarly defined in terms of negative numbers in the auxiliary sequence. The sequence $h_1, h_2, \cdots, h_n$ is ordinarily obtained as a permutation of a sequence $a_1, a_2, \cdots, a_n$, where the terms of the latter sequence are independent observations of some stochastic variable. When this is the case, then r_p (the number of runs both up and down of length p) and r_p' (the number of runs of length at least p) are also stochastic variables. In applying this theory it is important to know the means $E(r_p)$ and $E(r_p')$, the variances $\sigma^2(r_p)$ and $\sigma^2(r_p')$ and the covariances $\sigma(r_p r_q)$ and $\sigma(r_p' r_q')$. The authors perform these computations for the case in which the original stochastic variable has an equal probability distribution in the interval 0,1. The results of these computations serve to point out certain current misconceptions concerning runs up and down.

A. H. Copeland (Ann Arbor, Mich.).

Grubbs, Frank E. On the distribution of the radial standard deviation. Ann. Math. Statistics **15**, 75–81 (1944). [MF 10240]

When samples of bullets are to be measured for accuracy by their scattering on a target, a knowledge of the distribution function of the sample radial standard deviation $Z=(s_x^2+s_y^2)^{\frac{1}{2}}$ is desirable. The author's main purpose is to obtain a certain series expansion for the distribution function $dF(Z)$ assuming the bivariate universe is normally distributed. When the universe σ_x^2, σ_y^2 are equal, $dF(Z)$ is obtained immediately and in finite form by compounding two χ^2 distributions but the author gives also an independent derivation using characteristic functions. When $\sigma_x^2 \neq \sigma_y^2$, the characteristic function of Z^2 is constructed and then exhibited as a convergent infinite series, whence termwise Fourier integration gives $dF(Z^2)$ in series form. For use in significance tests, Prob. $\{Z^2 < k^2(\sigma_x^2+\sigma_y^2)\}$ is expressed as an infinite series of incomplete gamma functions. Finally for the important case $\sigma_x=\sigma_y$ a table is appended for samples up to $N=15$ giving the mean and standard deviation of the Z-distribution together with values of k at four important percentage points.

J. L. Vanderslice.

Bose, Purnendu and Raja Rao, S. On the limiting forms of statistical distributions. Science and Culture **9**, 402–403 (1944). [MF 10279]

Irwin, J. O. A table of the variance of $\sqrt{x}$ when x has a Poisson distribution. J. Roy. Statist. Soc. (N.S.) **106**, 143–144 (1943). [MF 9916]

The table gives the variance of $x^{\frac{1}{2}}$ when x has a Poisson distribution with mean μ; the values are given for $0 \leqq \mu \leqq 15$ in steps of .1 and to four decimal places.

W. Feller.

Feller, W. On a general class of "contagious" distributions. Ann. Math. Statistics **14**, 389–400 (1943). [MF 9844]

A general process of compounding a c.d.f. (cumulative distribution function) $G(x)$ from a family $\{F(x|\theta)\}$ of c.d.f.'s by means of another c.d.f. $U(x)$ is defined by the formula $G(x)=\int F(x|\theta)dU(\theta)$. Here θ is a parameter, and $U(\theta)$ is such that it assigns zero probability to any domain of θ for which $F(x|\theta)$ is undefined. Then $G(x)$ is called a compound Poisson distribution if $F(x|\theta)$ is a Poisson c.d.f. with mean $\theta>0$; $G(x)$ is a generalized Poisson distribution if $F(x|\theta)$, defined for $\theta=1, 2, 3, \cdots$, is the θ-fold convolution of an arbitrary c.d.f. and $U(x)$ is a Poisson c.d.f. Further specialization in these two cases leads to various distributions in the literature, among them the Pólya-Eggenberger and certain ones introduced by Neyman for entomological applications. Among many interesting results is the following. If we compare a compound Poisson with an ordinary Poisson c.d.f. having the same mean, the former in general [precise statement by author] gives a higher probability to the category of no events. Of interest to the statistician should be the discussion of contagion and the conclusion that "an excellent fit of Pólya's distribution to observations is not necessarily indicative of any phenomenon of contagion in the mechanism behind the observed distribution."

H. Scheffé (Princeton, N. J.).

Herdan, G. The logical and analytical relationship between the theory of accidents and factor analysis. J. Roy. Statist. Soc. (N.S.) **106**, 125–142 (1943). [MF 9915]

Albert, A. A. The matrices of factor analysis. Proc. Nat. Acad. Sci. U.S.A. **30**, 90–95 (1944). [MF 10191]

The author considers the following problem arising in factor analysis. Given a symmetric matrix M_0 whose diagonal elements are 0, it is required to find a diagonal matrix D such that $M=M_0+D$ has minimum rank ρ. The ideal rank r of M_0 is defined as the highest order of a nonvanishing minor of M_0 formed from nondiagonal elements. Obviously $\rho \geqq r$. The author gives a method for finding D in the important special case $\rho=r$ and illustrates his method by an example.

H. B. Mann (Barrytown, N. Y.).

Epstein, Benjamin and Churchman, C. West. On the statistics of sensitivity data. Ann. Math. Statistics **15**, 90–96 (1944). [MF 10242]

An important complex of statistical problems that needs further mathematical analysis arises in many different fields of experiment; it is perhaps best known under the guise of "biological assay." The main problems [not stated in the paper] may be formulated as follows. We are given data consisting of n number triples (x_i, N_i, p_i), where p_i is the observed proportion of individuals affected when a dose of strength x_i is administered to N_i individuals. If we assume a monotone nondecreasing function $\pi(x)$ giving the binomial probability $\pi_i=\pi(x_i)$ that a random individual will be affected by a dose x_i, then the problems consist of making statistical inferences about the unknown function $\pi(x)$. Now $\pi(x)$ has the mathematical form of a cumulative distribution function, and the main purpose of this paper seems to be to estimate moments of $\pi(x)$ and to calculate the variances of these estimates. It is not clear what is to be done with the estimated moments, since the authors wish to avoid the usual assumption that the functional form of $\pi(x)$ is known. After a page of introduction, the analysis to follow is still unmotivated; where unknown parameters should appear in many of the formulas we find instead random variables; nowhere in the notation is the distinction made between a population parameter and its estimate from the sample; differentials are used for quantities not necessarily small and no hint given that the results may be only asymptotically correct.

H. Scheffé (Princeton, N. J.).

Bartky, Walter. Multiple sampling with constant probability. Ann. Math. Statistics **14**, 363–377 (1943). [MF 9842]

The author deals with multiple acceptance sampling from an infinite lot in which the first sample consists of n_0 units and subsequent samples, unlimited in number, contain n units each. For the first sample the acceptance number is c

and the rejection number is $c+k$; for the rth additional sample these numbers are, respectively, $c+r$ and $c+r+k$. If a sample neither accepts nor rejects the lot, another sample is taken. Using an interesting matrix formulation, it is shown that a lot will be eventually accepted or rejected, the probability that a lot will eventually be accepted is found and the expected number of additional samples is obtained. The latter two results are given in terms of coefficients of two power series, the means for the evaluation of which is developed. Special cases are examined and numerical approximations are found. There is an extensive table of results for particular cases. In the case of two kinds of lots with fractions defective p^* and p, $p^*<p$, if it is desired that the better lots always be accepted and the poorer ones very rarely, an elegant result on the minimum inspection of good lots is obtained. *C. C. Craig.*

Hansen, Morris H. and Hurwitz, William N. On the theory of sampling from finite populations. Ann. Math. Statistics 14, 333–362 (1943). [MF 9841]

This apparently is a further and more extended report of investigations carried on within the Bureau of the Census which were the subject of a brief paper by the authors [J. Amer. Statist. Assoc. 37, 89–94 (1942)]. The general subject is what has been learned about the design of stratified sampling schemes in view of existing administrative and cost restrictions and in the light of information gained concerning the relative magnitude of variances and covariances between and within various types of sampling units. Once a sampling scheme is settled upon, the method of estimation for maximum efficiency must be studied and here the authors point out that if the sampling units chosen from the strata are of varying size a nonlinear estimate, biased but consistent, may have a smaller mean square error than the best linear unbiased estimate, and examples of the former type are given and discussed. The main body of the paper has to do with gains introduced by the use of enlarged primary sampling units within strata, by the sampling of primary units with probabilities in proportion to their sizes, and the use of area substratification, for the description of which the reader is referred to the paper. The use and properties of estimates of ratios in the case of the application of all three of these principles to form "the specified subsampling system" is discussed. In the final section examples are given of gains in efficiency by the introduction of these principles and in the appendix algebraic derivations are given for the expected values and variances with comparisons of their magnitudes. *C. C. Craig.*

Mood, Alexander M. On the dependence of sampling inspection plans upon population distributions. Ann. Math. Statistics 14, 415–425 (1943). [MF 9846]

Assuming that the production process is statistically controlled in the sense that there exists a probability $P(N, X)$ that a lot of size N will contain X defective items, this paper investigates the relation of lot sampling inspection plans to the probability distribution $P(N, X)$. It is shown that: (1) the number of defective items in a sample from a lot is positively or negatively correlated with the number of defective items in the remainder of the lot according as the population distribution $P(N, X)$ is "flatter" than or "sharper" than a binomial distribution; (2) the sample can furnish no basis for an inference about the remainder of the lot if the population distribution is the binomial one; (3) the correlation between the number of defective items in the sample and the number of defective items in the corresponding original lot is positive. The importance of these results in setting up a sampling plan is briefly indicated. *W. A. Shewhart* (New York, N. Y.).

Bose, Chameli. Note on the sampling error in the method of double sampling. Sankhyā 6, 329–330 (1943). [MF 9081]

In the method of double sampling, two variates x, y, presumed to be correlated, are measured for each of a sample of n items from a population. The variate x is also measured for each item in an additional sample of N items. From the first sample, the linear regression of y on x is calculated and used to adjust the mean value of y to the expected value Y which corresponds to the mean value of x in the second sample. In certain circumstances the increase in accuracy obtained by means of the adjustment may more than repay the cost of the second sample. The sampling variance of Y includes (i) a component arising from errors in the estimated regression line and (ii) a component arising from errors in the mean $\bar{x}'$ of the second sample. Assuming that the samples are random and independent and that x and y follow a bivariate normal distribution, Bose gives three expressions for the sampling variance of Y: (1) taking account of both components of error; (2) taking account only of the second component; (3) taking account only of the first component. Corresponding expressions are given for the variance of $(Y-\bar{y}')$, where $\bar{y}'$ is the unknown mean value of y in the second sample. *W. G. Cochran.*

Bose, Chameli. The variance of the forecasted mean value subjecting to two-way fluctuations. Science and Culture 7, 514 (1942). [MF 9856]

An erroneous formula is given for the variance of the estimate Y in double sampling [see the preceding review] when both components of error are included. The first term in $1/n$ should be multiplied by $(1-2\rho^2)$. A formula given for the variance of $(Y-\bar{y}')$ is also incorrect, owing to the omission of a term $(-2\rho^2)$ inside the { } bracket. The author corrects these errors in a later paper [see above]. *W. G. Cochran* (Princeton, N. J.).

Madow, William G. and Madow, Lillian H. On the theory of systematic sampling. I. Ann. Math. Statistics 15, 1–24 (1944). [MF 10236]

If the items in the population are x_i $(i=1, \cdots, N)$, a systematic sample S_i of n items is defined as $\{x_{i+(\alpha-1)k}\}$, where $\alpha=1, \cdots, n$. The particular S_i is selected at random from the k possible samples, where $N=nk$. The notation is rather complicated and the proofs usually long; however, the results are of much practical value, since in many sampling projects, such as crop surveys, systematic sampling has been used and probably will be extended. One must be cautious in reading the article because $\bar{x}$ or $\hat{x}$ represents the systematic sample mean ($\bar{x}$ for a single S_i and $\hat{x}$ for a sample of g S_i's), while $\bar{x}$ represents the population mean. In all types of sampling investigated in this article, $\bar{x}$ or $\hat{x}$ is an unbiased estimate of $\bar{x}$. Also,

$$\sigma_{\bar{x}}^2=(\sigma^2/n)(1+2\sum \rho_{k\mu}),$$

where

$$nk\sigma^2\rho_{k\mu}=nkC_{k\mu}=\sum_{j=1}^{nk}(x_j-\bar{x})(x_{j+k\mu}-\bar{x})$$

and $nk\sigma^2=\sum_j(x_j-\bar{x})^2$ ($x_h=x_{h-nk}$ and $\mu=1, \cdots, (n-1)/2$ or $n/2$). Here $\rho_{k\mu}$ is called the circular serial correlation coef-

ficient; $\sigma_{\bar{x}}^2$ is also given in terms of the noncircular correlation and the serial variance.

Unbiased estimates of σ^2 usually can be obtained only if there are at least two systematic samples; the estimate r_{k_μ} of ρ_{k_μ} can not be unbiased for the finite population case which is considered here, but the ratio $(1-\rho_{k_\mu})/(1-\rho_{k_\mu}')$ is almost the expected value of $(1-r_{k_\mu})/(1-r_{k_\mu}')$. When $\bar{x}$ is known, σ^2 can be estimated from a single sample, but this estimate has but one degree of freedom. The authors state that $s_{\bar{x}}^2=\sum_\beta(\bar{x}_\beta-\bar{x})^2$, where g S_i's are selected at random from the total of k, is an unbiased estimate of $\sigma_{\bar{x}}^2$. Actually, for the given definitions of $\bar{x}$ and $\tilde{x}$,

$$E(s_{\bar{x}}^2)=k(g-1)\sigma_{\bar{x}}^2/g(k-1);$$

hence, the formulas for the unbiased estimates of σ^2 and C_{k_μ} must be changed.

If the population is divided into strata and samples are drawn at random from each stratum, the systematic sampling method of drawing the sample in the same order from each stratum is more efficient than that of reordering for each stratum if corresponding items in different strata are negatively correlated and inferior if they are positively correlated. In general, systematic sampling is more efficient than completely random sampling if $\sum\rho_{k_\mu}<-1/(2k)$ and n is large relative to k; random sampling is better if the x's have a period of k while systematic sampling generally is better if the period is $2k$. If the x's are distributed so that $x_j=j$, systematic sampling is superior to stratified random sampling for $N>2L^2-L$ and $c>L/(1-(L(L-1)/N))$, where there are L strata and c items are sampled from each stratum so that $Lc=nk$. Formulas also are given to determine when systematic sampling is more efficient for the special case $x_j=A+Bj+\varphi_2(j)$, where $\varphi_2(j)$ is a periodic function. *R. L. Anderson* (Princeton, N. J.).

Scheffé, Henry. Statistical inference in the non-parametric case. Ann. Math. Statistics 14, 305–332 (1943). [MF 9840]

This is a critical survey, thorough and with an exhaustive bibliography. After a short historical introduction and a formulation of the problem, the author discusses the existence of similar regions and the general randomization method (Fisher, Pitman, Scheffé). The method of ranks is described as a special case of the latter. The difficulties caused by "fitting" distributions and by ties are explained. Thereupon follow brief but complete descriptions of many of the nonparametric tests in the literature, too numerous to detail here. The author then discusses the estimation of certain numbers determined by an unknown distribution function (Pitman, Thompson), and of the unknown distribution function itself (Kolmogoroff, Smirnoff, Wald and Wolfowitz). A description of tolerance limits (Wilks, Wald) follows.

The last section of the paper is headed "Toward a General Theory" and takes up the criterion of consistency (Wald and Wolfowitz), the nonparametric likelihood ratio (Wolfowitz) and the general formulation of the problem of statistical inference by Wald. *J. Wolfowitz.*

Wald, A. and Wolfowitz, J. An exact test for randomness in the non-parametric case based on serial correlation. Ann. Math. Statistics 14, 378–388 (1943). [MF 9843]

If $x_1, x_2, \cdots, x_N$ is a sample, the hypothesis of randomness (of importance in the theory of quality control and also in the analysis of time series) is the following: $x_1, \cdots, x_N$ are independently distributed with a common cumulative distribution function $F(x)$, which is unspecified except perhaps for continuity. To test the hypothesis, the randomization idea is used with $R=x_Nx_1+\sum_{i=1}^{N-1}x_ix_{i+1}$ as statistic. This is equivalent to a test based on the serial correlation coefficient (circular definition) with lag h if h is prime to N. If F is continuous the proposed test is exact; if F is discontinuous but all its moments are finite and its variance is positive, the test is asymptotically exact. Besides calculating the mean and variance of the statistic R, the authors prove under mild conditions that its asymptotic distribution is normal. The latter result is especially interesting since it is the first of its kind for the case of the randomization method not based on the use of ranks. *H. Scheffé.*

Gumbel, E. J. On the plotting of flood-discharges. Trans. Amer. Geophys. Union **1943**, 699–719 (1943). [MF 10174]

Samuelson, Paul A. A fundamental multiplier identity. Econometrica 11, 221–226 (1943). [MF 9251]

This paper discusses two definitions of dynamic spending multipliers given in the literature. According to the first definition the multiplier is given by the level of income finally reached as a result of a constant autonomous increment in spending repeated in every period of time, while according to the second definition the multiplier is given by the cumulative sum of all the increments of income (summed over all time periods) generated by a single non-repeated impulse of expenditure. The author shows that these two definitions are identical. *A. Wald.*

Lange, Oscar. The theory of the multiplier. Econometrica 11, 227–245 (1943). [MF 9252]

The multiplier is a concept introduced in economic theory and measures the final increment in the value of an economic variable produced by an initial increment of another variable. In this paper the main types of multipliers are surveyed and their correct interpretations given, thus clearing up some of the misunderstandings and confusions which have arisen in this connection. The last two sections of the paper are devoted to a discussion of the theory of dynamic multipliers which takes into account the time necessary to pass from one equilibrium position to another and describes the path of transition. According to this dynamic theory the multiplier effect is a function of time which satisfies a linear difference or integral equation. *A. Wald.*

TOPOLOGY

McKinsey, J. C. C. and Tarski, Alfred. The algebra of topology. Ann. of Math. (2) 45, 141–191 (1944). [MF 9838]

The authors consider the properties of the Boolean operations $\cap$, $\cup$, $'$ and the closure operation Cx as applied to subsets x of a general topological space. Among their most fundamental results are the following. (1) Every abstract "closure algebra" is isomorphic with a subalgebra of the closure algebra of all subsets of some topological space. (A "closure algebra" is a Boolean algebra with a closure operation which satisfies $x\leqq Cx$, $CCx=Cx$, $C(x\cup y)=Cx\cup Cy$, $C\Lambda=\Lambda$. The proof of the result cited involves

Stone's representation theorem for Boolean algebras.) (2) If f and g are "closure-algebraic" functions (that is, functions definable from the four operations $\cap$, $\cup$, $'$, C), and if $Cf \cap Cg = \Lambda$ identically, then either $f = \Lambda$ or $g = \Lambda$. (3) If a closure-algebraic function f vanishes on every finite closure algebra, then it vanishes identically. (4) If S is any normal, dense in itself topological space with countable basis (for example, Euclidean n-space), then every finite closure algebra is isomorphic with a subalgebra of the (relative) closure algebra over an open subset of S. (It is a corollary of results (3)–(4) that any functional identity on the operations $\cap$, $\cup$, $'$, C which holds on the real line holds in the most general topological space, The authors show further that all "sentences" of a more general type involving only the operations $\cap$, $\cup$, $'$, C which hold in the Cantor discontinuum hold in every topological space.) (5) In any closure algebra generated by a finite number of elements, there is an identity which is not an identity for closure operations in all topological spaces.

Many other interesting new ideas and results may be found, including the abstract concepts of "functionally free" and "absolutely free" algebras, which apply to any abstract algebra. *G. Birkhoff* (Cambridge, Mass.).

Colmez, Jean. Sur le problème de Wiener: solution générale du problème dans le cas le plus général. C. R. Acad. Sci. Paris **215**, 519–521 (1942). [MF 10185]

The problem considered is that of topologizing a set P of elements so that each of a collection σ of permutations is a homeomorphism of the topologized space P. If G is the smallest group of permutations containing σ, all possible Kuratowski topologies [see Fund. Math. **3**, 182–199 (1922)] are obtained by the following construction for the family of sets closed in P. Beginning with any family of sets which includes the null set and the set P, adjoin successively all transforms of elements by members of G, all finite sums of transforms and all intersections of sums. If P is nonfinite there always exist nontrivial topologizations. If G is the group of all permutations of P, the possible topologizations can obviously be classified by the powers of the closed sets. If the (nontrivially) topologized space is to be a Hausdorff space certain restrictions must be put on G, for example, G cannot contain the group of all finite permutations of P. *J. L. Kelley* (Aberdeen, Md.).

Day, Mahlon M. Convergence, closure and neighborhoods. Duke Math. J. **11**, 181–199 (1944). [MF 10158]

A set X is called a closure space if a closure function c is given associating with each subset A of X another subset cA of X. A set X is called a neighborhood space if with each element $x \in X$ there is associated a family $N(x)$ of subsets of X. An ordered system $\mathfrak{A} = (A, >)$ consists of an arbitrary set A and an arbitrary binary relation $>$ in A. The set X is called a limit space with respect to a family $\{\mathfrak{A}\}$ of ordered systems for every $\mathfrak{A} = (A, >)$ in $\{\mathfrak{A}\}$ and for every function f mapping A into X a subset Lf of X is given. The author shows that each closure function c leads in a natural fashion to a neighborhood function N_c and a limit function L_c. Similarly each neighborhood function N leads to a closure c_N and to a limit L_N. Finally a limit function L defines a closure c_L and a neighborhood function N_L. This way L_{N_L} will denote the limit space corresponding to the neighborhood space N_L. In general relations of the type $L = L_{N_L}$ are not true. However, if the family $\{\mathfrak{A}\}$ contains only transitive systems the author shows that the total number of spaces that can be obtained from a given c, N or L by iteration of the above operations is finite and can be exhibited on very instructive diagrams. *S. Eilenberg.*

White, Paul A. On r-regular convergence. Bull. Amer. Math. Soc. **50**, 123–128 (1944). [MF 9897]

The idea of r-regular convergence was introduced by G. T. Whyburn [Fund. Math. **25**, 408–426 (1935)] to study the limits of sequences of arcs, simple closed curves, 2-cells and topological spheres for the cases $r = 0$, $r = 1$. The present paper extends some of Whyburn's results to higher dimensions, although the lack of simple characterizations of the higher dimensional sets makes the results much weaker than for the simpler cases. The definitions are phrased as usual in the combinatorial language of Vietoris cycles and chains having nonoriented cycles and chains as their coordinates. The following theorems are typical of the results obtained. Let M_i converge s-regularly to M for all s not exceeding a certain integer k. Then (1) if $k = r-1$ and C is the smallest carrier of an essential projection cycle of M, then C can be expressed as the limit of a sequence C_i, each set of which is the smallest carrier of some cycle of M_i and these cycles are essential for all sufficiently large i. (2) If $k = r$ and B is an irreducible membrane for an homology in M, then B can be expressed as the limit of a sequence of sets B_i, where B_i is a subset of M_i and is an irreducible membrane for an homology in this set. (3) If $k = r$ and each M_i is a closed r-dimensional Cantorian manifold, and if the dimension of M does not exceed r, then M is also a closed r-dimensional Cantorian manifold. The results (1) and (2) just stated have the following interesting corollaries in the language of the cyclic elements of higher order [G. T. Whyburn, Amer. J. Math. **56**, 133–146 (1934)]: (1′) if each M_i in (1) is a T_r-set, then so also is M; (2′) in (2) if A_i is an A_r-set of M_i for each i and if A_i converges to A, then A is an A_r-set of M. *D. W. Hall.*

Otchan, G. L'espace des sousensembles d'un espace topologique. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 340–352 (1943). (Russian. French summary) [MF 10228]

The author presents here a detailed exposition of results previously announced without proof [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 107–109 (1941); these Rev. **3**, 138]. *J. V. Wehausen* (Columbia, Mo.).

Harrold, O. G., Jr. The construction of a certain metric. Duke Math. J. **11**, 23–34 (1944). [MF 10144]

In a previous paper [S. Eilenberg and O. G. Harrold, Amer. J. Math. **65**, 137–146 (1943); these Rev. **4**, 172] conditions were given characterizing those continua having a finite linear measure upon the choice of an appropriate metric. The present paper constructs such a metric in which the neighborhood system of the continuum plays the fundamental role and which has the added advantage of being convex. The main result of the paper states that for a continuum X the following conditions are equivalent. (1) To each point p of X and $\epsilon > 0$ there exists an uncountable family of neighborhoods $[U_\alpha]$ of p each of diameter less than ϵ, each with a boundary which is a finite set of points, and such that for arbitrary U_α, U_β in $[U_\alpha]$ one of these neighborhoods contains the closure of the other. (2) There is a convex metric under which X has a finite linear measure and which is topologically equivalent to the given metric in X. The general plan of the proof is to approximate X by a monotone increasing family of stably regular continua

and then extract the desired metric for X from the corresponding metrics of these continua. The author establishes the following characterization of continua of finite degree which is useful in the proof. In order that the continuum X be of finite degree it is necessary and sufficient that to $\epsilon>0$ there corresponds a decomposition $X=X_1+X_2+\cdots+X_n$, where each X_i is a continuum of diameter less than ϵ and for i not j the product $X_i \cdot X_j$ is a finite subset of the points of degree 2 in X. *D. W. Hall.*

Sorgenfrey, R. H. Some theorems on co-terminal arcs. Bull. Amer. Math. Soc. **50**, 257–259 (1944). [MF 10208]

The author shows that, if H, K and L are three distinct arcs from A to B, then there are two of them neither of which is a subset of the sum of the other two. Moreover, if H, K and L are three arcs emanating from a point A and no one of them is a subset of any other, then there is one of them which fails to be a subset of the sum of the other two. The validity of this latter theorem answers a question raised by Harlan C. Miller. An example constructed by the familiar "canal" method shows that neither of the above theorems is true if the word "arcs" in the hypothesis is replaced by the words "irreducible continua."

D. W. Hall (College Park, Md.).

García, Mariano. Component orbits under pointwise recurrent homeomorphisms. Bull. Amer. Math. Soc. **50**, 260–266 (1944). [MF 10209]

The notion of a component orbit is due to G. E. Schweigert [Bull. Amer. Math. Soc. **46**, 963–969 (1940); these Rev. **2**, 179] and is closely associated with two theorems established by D. W. Hall and G. E. Schweigert [Duke Math. J. **4**, 719–724 (1938)], but so far has been considered only in connection with pointwise periodic homeomorphisms. The present paper considers an extended concept of this notion and, by applying methods analogous to those used by G. T. Whyburn [Analytic Topology, Amer. Math. Soc. Colloquium Publ., vol. 28, New York, 1942, pp. 259–261; these Rev. **4**, 86] in obtaining the results of Hall and Schweigert, exhibits certain more general theorems which contain these results as special cases.

The author considers a separable metric space X, a homeomorphism $f(X)=X$ defined on X, and denotes by x an arbitrary point of X. The point x is recurrent under f provided that given any neighborhood U of x there is a positive integer n such that $f^n(x)$ lies in U. The mapping f is pointwise recurrent on a subset A of X if each point of A is recurrent under f. The set-orbit of a subset A of X consists of the union of all sets of the form $f^n(A)$, where n ranges over all positive and negative integers and zero. A nonempty invariant set L is a component orbit provided L is the set orbit of one of its components. A component orbit one of whose components is invariant under a positive power of f is called a periodic component orbit.

The following are the results of the paper. (1) If X is compact and $\{G_n\}$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q then (a) if both f and its inverse are pointwise recurrent on $\limsup G_n - Q$, then $\limsup G_n$ is a periodic component orbit, (b) if f is pointwise recurrent on $\limsup G_n + \sum_1^\infty G_n$, then $\limsup G_n$ is a periodic component orbit. (2) Under the hypotheses of (1a) or (1b) the number of components of $\limsup G_n$ can not exceed the number of components of Q. Thus, if $\liminf G_n$ contains an invariant connected set, then $\limsup G_n$ is connected. (3) Let X be compact and G a component orbit in X. If f is pointwise recurrent on $\bar{G}$ and $\bar{G}$ contains a periodic component orbit Q, then $\bar{G}$ is a periodic component orbit and the number of its components does not exceed the number of components of Q. (4) Any pointwise periodic continuous mapping g of a compact totally disconnected space X into itself has equicontinuous powers. (5) If X is compact and $L=L_1+L_2$ is a separation of the nonempty closed invariant set L in X such that f is pointwise recurrent on L and, for $i=1$ or for $i=2$, the set L_i has the property that the closure of the orbit of each of its points is a periodic component orbit, then there exists a positive integer N such that $f^N(L_i)=L_i$, $i=1, 2$. *D. W. Hall* (College Park, Md.).

Gottschalk, W. H. Powers of homeomorphisms with almost periodic properties. Bull. Amer. Math. Soc. **50**, 222–227 (1944). [MF 10201]

Let X be a topological space and f a homeomorphism $X \to X$. A point x of X is called recurrent under f if it is carried into arbitrarily small neighborhoods of itself by suitable powers of f. If x is carried into a fixed but arbitrary neighborhood of itself by f^{n_i} with $n_1<n_2<\cdots$, $n_{i+1}-n_i$ uniformly bounded, it is called almost periodic. The author studies the effect of replacing f by a power of itself: the property of being a recurrent point is preserved, as is (if X is compact metric) the property of being an a. p. point. If X admits closed proper subsets which are invariant under f^n but none which are invariant under f, then, for suitable divisor n of $|k|$, X admits a decomposition into n closed sets which are invariant under f^n and are minimal in this respect. (In this case, X can not be connected.) If X is metric and admits a decomposition consisting of closures of orbits of f, then each point is a. p. under f. The converse is true if X is also compact. *P. A. Smith.*

Whyburn, G. T. Homotopy reductions of mappings into the circle. Duke Math. J. **11**, 35–42 (1944). [MF 10145]

A contribution to the study of continuous mappings of locally connected continua into the circle K. The principle result is that, if $f: A \to K$ is such a mapping, then f is homotopic to a mapping φ such that, if M is a continuum in $\varphi(A)$ with nonempty interior, then $\varphi^{-1}(M)$ has just a finite number of components, and each of the latter is mapped by φ onto M. *R. L. Wilder.*

de Groot, J. On the extension of continuous functions. Nederl. Akad. Wetensch., Proc. **45**, 842–843 (1942). [MF 10440]

The author proves the elementary fact that, if A is a nonclosed subset of a metric space M, then it is impossible to extend every continuous (bounded) real function on A to a continuous (bounded) real function on M. He remarks that the theorem is not true for normal spaces. [Cf. the following review] *D. Montgomery* (Princeton, N. J.).

de Groot, J. Bemerkung zum Problem der topologischen Erweiterung von Abbildungen. Nederl. Akad. Wetensch., Proc. **45**, 655–657 (1942). [MF 10427]

If A is a nonclosed or a nonbounded subset of R^n then there exists a homeomorphism taking A into a subset of R^n with the property that this homeomorphism can not be extended to R^n. The proof is elementary.

D. Montgomery (Princeton, N. J.).

Alexandroff, Paul. Allgemeiner Dualitätssatz für Projektionsspektra und für im kleinen bikompakte Räume. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 315–319 (1941). (Russian. German summary) [MF 10290]

A preliminary report of a paper, since published [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 227–282 (1942); these Rev. **4**, 249; also Trans. Amer. Math. Soc. **54**, 286–339 (1943); these Rev. **5**, 48].

S. Eilenberg (Ann Arbor, Mich.).

Ehresmann, Charles. Sur les espaces fibrés associés à une variété différentiable. C. R. Acad. Sci. Paris **216**, 628–630 (1943). [MF 9989]

Earlier results of the author and J. Feldbau on fibre-spaces [same C.R. **212**, 945–948 (1941); **213**, 762–764 (1941); **214**, 144–147 (1942); these Rev. **3**, 58; **5**, 148; **4**, 146, respectively] are applied to the space of tangent linear vectors to a differentiable manifold V, considered as fibre-space with V as base and the affine space as fibre; an immediate consequence of these results is that there exists on V a quadratic ds^2 of given signature p if and only if there is on V a nonsingular field of nonoriented p-dimensional elements. For $p=1$, the dimension of V being 4, this gives necessary conditions for the topological structure of a relativistic universe (excluding, for example, the 4-sphere and the projective 4-space); there are additional conditions if past and future are to be distinguishable "in the large" (the Euler characteristic must be 0). For $p=0$, this shows that there always exists a positive-definite ds^2 on a differentiable manifold [the author is clearly unacquainted with the recent work of N. E. Steenrod, where, in particular, the last-mentioned result is also to be found [Ann. of Math. (2) **43**, 116–131 (1942); these Rev. **3**, 144]]. *A. Weil.*

Clark, C. E. The symmetric join of a complex. Bull. Amer. Math. Soc. **50**, 81–88 (1944). [MF 9890]

With every pair of points p,q of a polyhedron K there can be associated a closed line segment pq, with the convention that distinct segments do not intersect except at a common endpoint and that the length of pq is a continuous function of p and q which approaches zero if p and q approach a common limit. The points of these segments together with their obvious natural topology make up J, the symmetric join of K. The topological product of K by itself is denoted by P, while S stands for the symmetric product of K defined in terms of the midpoints A_{pq} of the segments pq. The "neighborhood" N_S, which has S as a homotopic retract, consists of all closed segments formed by going from the midpoint of the segment from p to A_{pq} to the midpoint of the segment from q to the same point. Similarly the "neighborhood" N_K, which has K as a homotopic retract, is defined as consisting of all closed segments of the type joining p to A_{pq}. It is shown that there is a simplicial division of J such that K, S, P, N_S and N_K carry subcomplexes of the division, the first of which is a subdivision of the given polyhedron K, and the second and third of which are subdivisions of natural cell divisions of P and S previously determined. The author then studies the homologies in P and S and applies the theory to the sum complex $J=N_S+N_K$. The group of the sum cycles, modulo those that bound in J, is also called S and is shown to be generated by a free 0-dimensional element and a set of elements of order 2, each containing exactly one cycle of a specified type. The paper concludes with discussion of the seams and Betti group of J. *D. W. Hall.*

Steenrod, N. E. The classification of sphere bundles. Ann. of Math. (2) **45**, 294–311 (1944). [MF 10267]

The reviewer has shown [Bull. Amer. Math. Soc. **43**, 785–805 (1937)] how sphere bundles (with spheres of dimention k) may be defined by mapping the base space B (assumed to be a complex) into the space M_l^k of great k-spheres of a $(k+l+1)$-sphere for some l. The author shows that two such bundles are equivalent if and only if the corresponding mappings are homotopic. The homotopy groups of M_l^k are computed for dimensions not greater than 6; this leads to a solution of the classification problem in case B is a sphere of dimension not greater than 6. In particular, there are an infinity of k-sphere bundles over S^4 if $k\geqq 2$ (this was known for $k=2$); a k-sphere bundle over S^3 (S^4) is always a product bundle if $k\geqq 5$ ($k\neq 5$).

H. Whitney (Cambridge, Mass.).

Chogoshvili, George. Behaviour of some topological invariants on level surfaces. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 995–999 (1942). (English. Georgian summary) [MF 10322]

Let M be a differentiable manifold and f a real-valued function of class C' defined over M with nondegenerate critical points. Let c be a critical value of f and $\epsilon>0$ such that $c-\epsilon\leqq f\leqq c+\epsilon$ contains just one critical point of index k. Let $L_p=E\ (f=c-\epsilon)$ and $L_q=E\ (f=c+\epsilon)$. Then L_q can be obtained from L_p by deleting from L_p the product Q of an $(n-k)$-cell with a $(k-1)$-sphere and inserting the product P of a k-cell with an $(n-k-1)$-sphere. If $I(F, M)$ is a function defined for closed sets $F\subset M$ with positive integer values satisfying (1) $F_1\supset F_2$ implies $I(F_1, M)\geqq I(F_2, M)$, (2) $I(F_1+F_2, M)\leqq I(F_1, M)+I(F_2, M)$, then the inequalities $-I(Q, M)\leqq I(L_q, M)-I(L_p, M)\leqq I(P, M)$ follow. In particular, the homotopy category $\mathrm{cat}_M F$ is such a function. This yields the result $\mathrm{cat}_M L_q-\mathrm{cat}_M L_p=0$ or ± 1. This in turn leads to cat L_q-cat $L_p=0$, ± 1 or ± 2, where cat L means $\mathrm{cat}_L L$. *N. E. Steenrod* (Washington, D. C.).

GEOMETRY

Blumenthal, L. M. Distance geometry notes. Bull. Amer. Math. Soc. **50**, 235–241 (1944). [MF 10204]

Three (not connected) facts of metric geometry are established. (1) Let R be a complete, convex, externally convex metric space. If L is any straight line in R, p a point not on L, p_0 a foot of p on L, and x an arbitrary point on L, let $pp_0^2+p_0x^2=px^2$. Then R is congruent with a Euclidean or Hilbert space. (2) The metric space R is said to have covering indices (s, t) with respect to the metric space R' if any subset α of R' containing at least $s+t+1$ points is congruently contained in R as soon as any s points of α have this property. It is shown that any n-dimensional sphere has covering indices $(n+2, 1)$ with respect to E^{n+1}. Whether these indices are best is left open, except for $n=1, 2$. (3) A comparatively simple proof is given for the following well-known fact. If a semimetric space S of more than $n+3$ points has each of its $(n+2)$-tuples congruently contained in E^n, then S is congruently contained in E^n.

H. Busemann (Chicago, Ill.).

* please see p. 328 Errata if interested in knowing the original definition of covering indices as given by the author.

Busemann, Herbert. On spaces in which two points determine a geodesic. Trans. Amer. Math. Soc. **54**, 171–184 (1943). [MF 9117]

Continuing his researches on abstract metric spaces with particular emphasis on the geodesics [Metric Methods in Finsler Spaces and in the Foundations of Geometry, Princeton University Press, Princeton, N. J., 1942; these Rev. **4**, 109], the author considers those spaces which satisfy, besides his four axioms, the further axiom that there is only one geodesic through any two distinct points of the space. The main theorem asserts that such a space possesses one of the properties: (a) it is simply connected with all geodesics congruent to Euclidean straight lines; (b) it has a sphere like space as two-sheeted universal covering space and its geodesics are congruent to Euclidean circles and have the same length. This result can be utilized to simplify and complete the theory of spaces with convex spheres. Finally, two-dimensional spaces are considered and those for which the geodesics are straight lines are determined in the sense of topology. *S. Chern* (Princeton, N. J.).

Blaschke, Wilhelm. Nicht-Euklidische Geometrie und Mechanik. I, II, III. Hamburger Math. Einzelschr. **34**, 82 pp. (1942). [MF 9806]

This pamphlet opens with a discussion of quaternions, which are used freely throughout, and a historical summary of elliptic geometry including Clifford parallels and Cayley's projective metric. The equations of kinematics and motion for elliptic space are then treated in a short chapter. The third chapter, about half the pamphlet, is devoted to differential geometry and the analogues of many classical theorems of Euclidean space are developed.

P. Franklin (Cambridge, Mass.).

Schwerdtfeger, H. Skew-symmetric matrices and projective geometry. Amer. Math. Monthly **51**, 137–148 (1944). [MF 10164]

The author shows how matrix theory may be used in some parts of projective geometry, particularly in the consideration of null systems. He shows that, if $A=P^{-1}Q$, where P and Q are two skew symmetric matrices of order $2m$ and P is nonsingular, A satisfies an equation of degree m, $f(x)=0$ ($f(x)$ is the square root of the characteristic function of A). He gives necessary conditions, in terms of the coefficients of $f(x)$, that two null systems with matrices P and Q be in involution, that is, that $P^{-1}Q=kQ^{-1}P$ or $A^2=kE$. These conditions cannot be sufficient since $A^2=kE$ implies not only that all characteristic roots of A have the value $\pm\sqrt{k}$ but also that all elementary divisors of $A-xE$ be linear. *J. Williamson* (Flushing, N. Y.).

Tietze, Heinrich. Verallgemeinerung einer Rekursionsformel für gewisse Polyeder-Volumina. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1942**, 17–20 (1942). [MF 9801]

The n-dimensional volumes of certain n-dimensional polyhedra in E_n are expressed in terms of the lower dimensional volumes of derived figures. *J. L. Dorroh.*

Durham, R. L. A simple construction for the approximate trisection of an angle. Amer. Math. Monthly **51**, 217–218 (1944). [MF 10257]

Kárteszi, Franz. Über das System der gleichseitigen Hyperbeln, die eine Parabel hyperoskulieren. Monatsh. Math. Phys. **50**, 27–34 (1941). [MF 10489]

Wayne, Alan. A table for computing perimeters of ellipses. Amer. Math. Monthly **51**, 219–220 (1944). [MF 10258]

Thébault, Victor. Sphères associées au tétraèdre. C. R. Acad. Sci. Paris **216**, 21–23 (1943). [MF 9996]

Given a tetrahedron $(T)=ABCD$ ($BC=a$, $CA=b$, $AB=c$, $DA=a'$, $DB=b'$, $DC=c'$) having G for centroid and 0, R for center and radius of the circumscribed sphere, let (ω, ρ) be the sphere for which the vertices A, B, C, D have, respectively, the powers kl^2, km^2, kn^2, kp^2, where l, m, n, p are given segments and k an arbitrary parameter. If the vertices A, B, C, D are kept fixed and the quantities kl^2, km^2, kn^2, kp^2 are subjected to circular permutations, three analogous spheres (ω', ρ'), (ω'', ρ''), (ω''', ρ''') are obtained, said to be "associated" to (ω, ρ).

Considering the particular case when $l^2=(a^2+b'^2+c'^2)/2$, $m^2=(a'^2+b^2+c'^2)/2$, $n^2=(a'^2+b'^2+c^2)/2$, $p^2=(a^2+b^2+c^2)/2$, the author proves that the following five spheres are coaxial: the circumsphere, the twelve point sphere, the sphere having for diameter the segment joining the centroid to the Monge point, the orthoptic sphere of the inscribed Steiner ellipsoid and the Longchamps sphere of the medial tetrahedron of the given tetrahedron. Some of these results have been anticipated [Court, Bull. Amer. Math. Soc. **48**, 583–589 (1942); these Rev. **4**, 51]. The paper has some annoying misprints. *N. A. Court* (Norman, Okla.).

Hameed, Asghar. On mutually self-polar simplexes. Bull. Calcutta Math. Soc. **35**, 43–53 (1943). [MF 9948]

The paper is an extension to n dimensions of results obtained by the author concerning self-polar tetrahedrons in three-dimensional space [same Bull. **33**, 157–186 (1941); these Rev. **4**, 111]. In the case of $n=4$ a quartic surface, in three dimensions, is brought to light having ten double points forming a three dimensional Desargues configuration 10_3. *N. A. Court* (Norman, Okla.).

Differential Geometry

Schmidov, F. I. Metrical contingents. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 247–249 (1943). [MF 9828]

Let E be a set of Euclidean 3-space, $\nu(X)$ a set function. If $S(p, \rho)$ denotes a spherical neighborhood of p with radius ρ, and $K(\xi, \theta, \rho)$ the interior of a half cone with p as vertex, ξ as axis and θ as vertical angle, then ξ is called a metric semitangent of E at p when

$$\underline{\lim}_{\rho\to 0}[\nu(E\cdot S(p,\rho)\cdot K(\xi,\theta,\rho))\rho^{-\tau}]>0$$

for all $\theta>0$; $0\leqq\tau\leqq 3$ seems to mean a fixed number depending on the set function ν. Let $E_{\alpha\beta}$ denote the set of all points in E with $\nu(E\cdot S(p,\rho))\cdot\rho^{-\tau}>\alpha$ for $\rho<\beta$. The main result is this: if ξ is a ray with origin p in $E_{\alpha\beta}$ and not a metric semitangent of E, then it is not a semitangent of $E_{\alpha\beta}$ at p in the sense of Bouligand. *H. Busemann* (Chicago, Ill.).

Humbert, Pierre. Sur certaines figures planes de l'espace attaché à l'opérateur Δ_2. Bull. Sci. Math. (2) **66**, 145–154 (1942). [MF 10472]

Cf. the author's notes in C. R. Acad. Sci. Paris **209**, 590–591 (1939); **211**, 530–531 (1940); these Rev. **1**, 172; **3**, 185.

Vincensini, Paul. Quelques remarques sur certaines caustiques par réflexion et sur le voisinage du second ordre d'un point d'une courbe. Bull. Sci. Math. (2) **66**, 156–166 (1942). [MF 10473]

Kerawala, S. M. A note on the affine evolute. Proc. Benares Math. Soc. (N.S.) 4, 39–40 (1943). [MF 10347]

Hopfner, F. Die Parallelkurven eines Büschels geodätischer Kurven des abgeplatteten Rotationsellipsoids. Z. Vermessungswesen 71, 153–176 (1942). [MF 9793]

This paper is devoted to a detailed study of the properties of the geodesic circles on an ellipsoid of revolution which is nearly a sphere. The equation of the ellipsoid is $x^2+y^2+z^2/c^2=1$, $c<1$, and let $\alpha=1-c$. The equations of the geodesics are then approximated by neglecting powers of α greater than one, the approximate equations are integrated and parametric equations of the (approximate) geodesic circles are obtained in terms of elementary functions. The properties of the geodesic circles, such as the presence of cusps and whether the geodesic radius yields a minimum of arc length, are then analyzed, chiefly for a radius of the geodesic circle which is nearly equal to π. *G. A. Hedlund.*

Löbell, Frank. Grundlinien einer differentiellen Theorie der Somenkongruenzen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1942, 1–16 (1942). [MF 9800]

E. Study's concept of soma (=moving trihedron) is studied from the point of view of differential geometry, the analogue of a surface being the Somenkongruenz (=2-parameter family of trihedra). The analytical tool used is the "screw" [R. Ball, F. Klein], which is represented as a dual vector $x=\bar{x}+\epsilon\bar{\bar{x}}$ ($\bar{x}$, $\bar{\bar{x}}$ vectors, $\epsilon^2=0$). Screws enter in the problem mainly as velocity screws, corresponding to an infinitesimal displacement of the soma. A Somenkongruenz, with parameters p and q, determines two screws, functions of p and q, corresponding to variations of p or q alone. Two screws, given in advance as functions of p and q, must satisfy a certain differential equation in order to determine a Somenkongruenz. This equation is equivalent to the well-known Darboux integrability conditions for two-parameter families of trihedra. It can be transformed into an equation connecting an integral over a domain in the p-q-plane with an integral over the boundary curve of the domain; this last equation is the "somatic" generalization of the Gauss-Bonnet formula, which can easily be deduced from it. *H. Samelson* (Syracuse, N. Y.).

Choudhury, A. C. On a generalisation of Reidemeister's figure in a web. Bull. Calcutta Math. Soc. 35, 77–80 (1943). [MF 9953]

This paper is a continuation of an earlier one [same Bull. 34, 93–98 (1942); these Rev. 5, 13]. Both papers deal with the geometry of 3-webs, but, where the previous paper discussed Thomsen's triangle, the present paper takes up the figure of Reidemeister. The difference between these two figures is similar to that between one-sided and two-sided figures. Reidemeister's figure is generalized to the case in which the edges are two hexagons, studied from the point of view of the relations in the corresponding quasi-group [H. Dibbert, Abh. Math. Sem. Hansischen Univ. 7, 323–355 (1937)]. The fact that this figure is closed corresponds to a relation $ad=bc$, which admits the transformation $da=cb$, though ad and bc may not be commutative. *D. J. Struik* (Cambridge, Mass.).

Cartan, Élie. Sur les couples de surfaces applicables avec conservation des courbures principales. Bull. Sci. Math. (2) 66, 55–72, 74–85 (1942). [MF 10465]

The author studies couples of surfaces in a three-dimensional Euclidean space which can be isometrically transformed into another surface so that the principal curvatures are conserved and transformation is neither a motion nor a reflection. In the complex domain O. Bonnet showed that this class of surfaces can be divided into three subclasses: (1) surfaces of constant mean curvature; (2) a class of surfaces of variable mean curvature having the same degree of generality as those of the first class; (3) the so-called surfaces of Bonnet, which depend only on arbitrary constants. The author restricted himself to the real domain and showed that the class (2) contains no real surface. As for the surfaces of Bonnet, it is natural to divide them into families such that each family consists of all surfaces which can be isometrically transformed to a given one with conservation of the principal curvatures. The number of surfaces in every such family, which are not congruent or symmetric to each other, is 1, 2 or 3. By a net of Bonnet is meant the net which corresponds to the net of lines of curvature of the other surface; they have various interesting geometrical properties. The method used in the treatment is that of moving trihedrals expressed in the simplified notation due to the author. *S. Chern.*

Doyle, Thomas C. Tensor theory of invariants for the projective differential geometry of a curved surface. Trans. Amer. Math. Soc. 55, 306–348 (1944). [MF 10217]

The aim of the paper is clearly indicated by its title. When a surface in a three-dimensional projective space is given by the parametric equations $y^4=y^4(u^1, u^2)$ in the homogeneous projective coordinates y^4, the projective differential invariants of the surface are functions constructed from $y^4(u^1, u^2)$, which are invariant under the following three groups: (1) the group of projective transformations of the space; (2) change of parameters; (3) multiplication of y^4 by a common nonzero factor. Starting from this remark, the author then proceeds with an elaborate treatment of the projective differential invariants, occasionally with their geometrical interpretations. A large part of the paper deals with known results expressed in terms of the notation of the author.

The author mentioned in this paper Kowalewski's method of "Identitätsbedingungen" for the geometry of a configuration under a Lie group, which is, at least in this particular case of a projective group, essentially the same as E. Cartan's method of moving frames. Following this general method, one gets the results of this paper in a much simpler way [cf. E. Cartan, Leçons sur la théorie des espaces à connexion projection, Gauthier-Villars, Paris, 1937]. *S. Chern* (Princeton, N. J.).

Grove, V. G. The transformation of Čech. Bull. Amer. Math. Soc. 50, 231–234 (1944). [MF 10203]

Let N be a conjugate net on a surface S of projective space. The line joining a point x of S to its canonical point with respect to N meets the reciprocal of the Fubini normal in a covariant point, the conjugal point of N at x. Any plane through this point is called a conjugal plane of N. A quadric having second order contact at x such that the polar of $(0, 0, 0, 1)$ is a conjugal plane is called a conjugal quadric. It is shown that a conjugal quadric may also be defined as any quadric having second order contact at x for which canonical lines of the first and second kind are reciprocal polars. As a by-product, new characterizations of the quadrics of Darboux and Davis and of certain canonical lines are obtained. The correspondence between a point on a tangent to a curve of N and its polar plane with respect to

a conjugal quadric of N has the form of a general transformation of Čech, the one parameter in this family of transformations depending on the choice of quadric. Thus one can say that the conjugal quadrics induce the general transformation of Čech and furnish a simple construction for the latter. Choices of the quadric which give certain important special cases of these transformations are indicated, for example, the polarity of Lie, the correspondences of Segre and of Moutard. *J. L. Vanderslice* (College Park, Md.).

Hsiung, Chuan-Chih. Plane sections of certain ruled surfaces associated with a curved surface. Duke Math. J. 11, 59–64 (1944). [MF 10147]

Let C be a curve through a point P on a nonruled surface S. As P describes C, the asymptotic tangents through P generate ruled surfaces R and R'. A plane π not through P intersects the asymptotic tangents at P in points T and T' on R and R', respectively. These latter points are ordinary points on the sections Γ and Γ' of R and R' by π. This paper studies these sections in some detail. The following results may be noted. As π revolves about the line l_2 joining T and T', the intersection of the tangents to Γ and Γ' at T and T' generate a line l_1 through P. The correspondence between l_1 and l_2 is a polarity with respect to a certain pencil Q of quadrics of Davis [W. M. Davis, Contributions to the general theory of conjugate nets, unpublished dissertation, Chicago, 1932]. These quadrics induce a particular transformation of Čech which the reviewer [cf. the preceding review] has called the associate correspondence of Segre. There is an unique pair of conics, one having second order contact with Γ at T, the other with Γ' at T'. The locus of each of these conics as π revolves about l_2 is a quadric belonging to the pencil Q. *V. G. Grove.*

Pa, Chenkuo. A generalization of associate quadrics of a surface. Amer. J. Math. 66, 115–121 (1944). [MF 9943]

Both B. Su [Tôhoku Math. J. 40, 433–448 (1935)] and S. Finikoff [Rec. Math. [Mat. Sbornik] 37, 48–97 (1930)] have defined an associate quadric of a nonruled analytic surface. Further, L. Godeaux [J. Chinese Math. Soc. 2, 1–5 (1937)] has shown the equivalence of this quadric and the second quadric in a Godeaux sequence. The object of the present paper is to generalize the definition of this quadric; the approach used is that of Finikoff.

Let C be a curve passing through the point M of a surface. At M and two consecutive points on C, there exist two systems of three asymptotic osculating linear complexes each of which have a regulus in common. The quadric, containing one of these reguli, is shown to be the associate quadric of Su when C is an asymptotic curve. Additional conditions are given such that the quadrics containing the previously mentioned reguli will be associate quadrics. Again, if all members of one family of reguli, touching a given tangent t of S at M have a straight line in common, then this line is called a characteristic line of S. Various properties of these characteristic lines are determined.

N. Coburn (Austin, Tex.).

Pa, Chenkuo. The projective theory of surfaces in ruled space. II. Amer. J. Math. 66, 101–114 (1944). [MF 9942]

In this paper the author continues the theory developed in a previous article [Amer. J. Math. 65, 712–736 (1943); these Rev. 5, 108]. First, the fundamental quantities associated with the asymptotic ruled surfaces R_1 and R_2 at the point P of the surface S are determined. Secondly, the author introduces a pencil of lines which characterize the second order invariants of the curve C_t of S. This is accomplished by determining the curved asymptotic lines C_1 of R_1 at P and C_2 of R_2 at P. The three principal points determined by this pair of curves are collinear and form the second principal line of C_t at P; the dual line in the correspondence of Bompiani forms the first principal line of C_t at P. With the aid of these concepts, it is shown that "the pangeodesic of a surface is characterized by the property that the first principal line at every point lies in the osculating plane at the point." Further results, involving the principal lines and the tangents of Segre, are obtained. By introducing the osculating linear complex of C_1 and working with Plücker coordinates, the author obtains results involving C_1 and C_2, the first principal line and directrix of the complex. Again, the author defines a system of curves C_ρ called deformable curves. It is shown that "the necessary and sufficient condition that two surfaces be projectively applicable is that the deformable curves, other than pangeodesics, of the same type on these surfaces be in correspondance." Furthermore, a polarity with respect to a pencil of quadrics is defined by means of C_1 and C_2, and its properties are studied. The paper concludes with a discussion of a sequence of asymptotic ruled surfaces associated with C_t. *N. Coburn* (Austin, Tex.).

Chern, Shiing-shen. Laplace transforms of a class of higher dimensional varieties in a projective space of n dimensions. Proc. Nat. Acad. Sci. U. S. A. 30, 95–97 (1944). [MF 10192]

This is a presentation of a method of extending the theory of conjugate nets and their Laplace transforms to r-dimensional varieties in projective n-space. Cartan's differential geometric methods are used. Proofs are omitted. Preliminary and essential to the extension the author gives a geometrical characterization of a class of varieties studied by Cartan [Bull. Soc. Math. France 47, 125–160 (1919); 48, 132–208 (1920)], which furnish an r-dimensional generalization of surfaces capable of supporting a conjugate net. He then finds that to each point P of a V_r of the above class $r(r-1)$ points P_{ij} can be invariantively associated. As P describes V_r each P_{ij} describes a variety of dimension not greater than r. Indeed, if the dimension is r the variety is also of the above class. It is these $r(r-1)$ varieties which are defined as the Laplace transforms of V_r, a definition which reduces to the classical one for $r=2$. A transform of a transform of V_r contains V_r as a component. The author concludes: "There is a great number of interesting problems which generalize the classical theory of periodic Laplace sequences. The problems seem to be particularly fascinating because it is probable to expect some results which are not direct generalizations of the theory of surfaces."

J. L. Vanderslice (College Park, Md.).

Devisme, Jacques. Sur quelques propriétés des trièdres d'Appell. C. R. Acad. Sci. Paris 212, 43–45 (1941). [MF 9182]

In a series of earlier papers [cf., in particular, J. Math. Pures Appl. (9) 19, 359–393 (1940); these Rev. 3, 21] the author has developed the geometry of the space whose element of arc is defined by $ds^3=dx^3+dy^3+dz^3-3dxdydz$. The object of this note is to study the same geometry by adopting E. Cartan's method of moving frames. A theorem on Appell trihedrals is stated. *S. Chern.*

Varga, O. Zur Herleitung des invarianten Differentials in Finslerschen Räumen. Monatsh. Math. Phys. **50**, 165–175 (1941). [MF 10480]

Cartan's Euclidean connection in a Finsler space is derived under a new set of conditions, which make use of Carathéodory's indicatrix and certain so-called osculating Riemannian spaces along one-parameter families of line elements. The results form essentially a new geometric interpretation of Cartan's connection. *S. Chern.*

RELATIVITY

de Beauregard, Olivier Costa. Sur deux questions de relativité. C. R. Acad. Sci. Paris **213**, 822–824 (1941). [MF 9659]

The author discusses two problems in special relativity. The first is concerned with the known fact that the magnetic moment can be represented by a tensor of the first or third rank. The second problem is trivial. *M. Wyman.*

Walker, A. G. Relativistic mechanics. III. Mechanics in 4-space, and conservative fields of force. Proc. London Math. Soc. (2) **48**, 161–179 (1943). [MF 9900]

The author has developed a theory of mechanics within the framework of kinematical relativity [Proc. London Math. Soc. (2) **46**, 113–154 (1940); these Rev. **1**, 283. The pp. 135–154 of that paper were intended as part II of the series.] basing himself on 3-dimensional space. This theory of mechanics is now extended to a 4-dimensional basis using the special "hyperbolic" 4-space characteristic of kinematical relativity. Two systems of coordinates in this 4-space are considered, the time-coordinates of which are related by Milne's relation

$$\tau = t_0 + t_0 \log (t/t_0).$$

Force, momentum, work and moments are defined, discussed in detail and reduced to forms which can be converted into any other coordinate systems permissible in the 4-space. The definition and properties of conservative fields of force and potential are also considered. The author concludes that there is in general no advantage in using the 4-space formulation for his theory of mechanics as against his original one in terms of 3-space. *G. C. McVittie.*

Einstein, A. and Bargmann, V. Bivector fields. Ann. of Math. (2) **45**, 1–14 (1944). [MF 9831]

Einstein, A. Bivector fields. II. Ann. of Math. (2) **45**, 15–23 (1944). [MF 9832]

These two papers present a mathematical discussion of a formalism for dealing with a generalization of tensors in which the various indices may refer either to points of different spaces or to different points of the same space. The new quantities are called bivectors. An example of such a quantity is the following. Let $A_1{}^i$ be a vector function of a point x_1 in R_1 and $A_2{}^i$ be a vector function of a point x_2 in R_2 (R_2 may be identical with R_1); then

$$T^{ik}_{12} = A_1{}^i B_2{}^k$$

in a second order bivector. The first paper discusses algebraic properties of bivectors and their behavior under various transformations, called rimming operations. It is mainly concerned with the equivalence of two symmetric bivectors under such a transformation. The second paper examines the relation between an affine connection (a mixed bivector) and a symmetric covariant one. *A. H. Taub.*

Barajas, Alberto. Birkhoff's theory of gravitation and Einstein's theory for weak fields. Proc. Nat. Acad. Sci. U. S. A. **30**, 54–57 (1944). [MF 10126]

In a recent review [G. D. Birkhoff, same Proc. **29**, 231–239 (1943); these Rev. **4**, 285] Weyl stated that the set-up for G. D. Birkhoff's theory of gravitation was much the same as a degenerate form of Einstein's theory. The author seems to misinterpret this comment to mean that the factual consequences of the two theories must be the same. For this reason a comparison is made of the following theories: (A) general relativity, (B) Einstein's theory of weak fields, (C) G. D. Birkhoff's theory. The results of the three crucial tests of relativity as given by (A) are taken to be correct and the same three tests are analyzed by means of (B) and (C).

The reviewer might point out that these analyses have been carried out before. In his original paper G. D. Birkhoff stated that (A) and (C) are in essential agreement, as far as the three crucial tests. Moreover, the fact that (A) and (B) disagree only with respect to the advance of perihelion has been known for some time [see Eddington, Mathematical Theory of Relativity, Cambridge University Press, 1923, p. 101]. Since Weyl undoubtedly knew of this disagreement and was merely remarking on the similarity in the set-ups of (B) and (C), this paper contributes nothing toward the refutation of Weyl's comments.

For the sake of those reading this paper it might be wise to correct several mistakes. The author copies a misprint that occurred in the original G. D. Birkhoff paper. Under the G. D. Birkhoff theory of gravitation the correct equations of motion of a free particle attracted by a spherical mass are

$$\ddot{x} = \frac{-mx}{r^3} - \frac{mx}{r^3}(\dot{x}^2+\dot{y}^2) \frac{+2m\dot{t}\dot{x}}{r^2},$$

$$\ddot{y} = \frac{-my}{r^3} - \frac{my}{r^3}(\dot{x}^2+\dot{y}^2) \frac{+2m\dot{t}\dot{y}}{r^2},$$

where the dot denotes differentiation with respect to "t." Moreover, when using the Einstein theory of weak fields, the proper line element is

$$ds^2 = (1-2m/r)dt^2 - (1+2m/r)(dx^2+dy^2+dz^2).$$

Thus the corrections to the gravitational potentials become $H_{ij} = (-2m/r)\delta_{ij}$, where δ_{ij} is the Kronecker delta. This leads to the equations of motion

$$\ddot{x} = \frac{-mx}{r^3} + \frac{4m\dot{t}\dot{x}}{r^2} - \frac{mx}{r^3}(\dot{x}^2+\dot{y}^2),$$

$$\ddot{y} = \frac{-my}{r^3} + \frac{4m\dot{t}\dot{y}}{r^2} - \frac{my}{r^3}(\dot{x}^2+\dot{y}^2).$$

However, even if the analysis had been carried out correctly it would only lead to known results and in no way would it refute Weyl's comments. *M. Wyman.*

Reichenbächer, Ernst. Die Erzeugung des Schwerefeldes. Z. Phys. **119**, 630–658 (1942). [MF 9412]

The author briefly refers to his "Foundations of a Theory of Matter" which is based on a general action principle and says he is going to introduce a principle of gauging. He then quotes a differential equation of the second order,

The reviewer would like to withdraw the claim that the author had copied a misprint which appeared in a paper by G. D. Birkhoff, and to say that the equations of motion obtained by using the Birkhoff theory of gravitation are correct as given in both papers when the differentiation symbol is taken to mean differentiation with respect to S. The equations given by the reviewer are correct when t is used as the independent variable.

giving its reference number on a page of an earlier work [Z. Phys. **107**, 285–309 (1937)] in which the formula is derived, and proceeds from that point. He assumes that in the metric

$$ds^2 = g_{11}dr^2 + r^2(d\theta^2 + \cos^2\theta d\varphi^2) - c^2 g_{44} dt^2$$

the product of g_{11} and g_{44} is unity. This leads him to a special solution for g_{11} and g_{44}. He then defines the mass in the field in terms of the total curvature of the 4-space. The motion of a free particle in the field is discussed at length and this leads to values for the deflection of light by the sun, for the motion of the perihelion of Mercury and for the gravitational reddening of light, which differ from Einstein's values, being 25% to 37% smaller for the deflection of light and the reddening. *G. C. McVittie.*

Petiau, Gérard. Sur la représentation unitaire de l'électromagnétisme et de la gravitation en mécanique ondulatoire. C. R. Acad. Sci. Paris **212**, 1126–1128 (1941). [MF 9189]

The author points out the connection between the equations describing a particle of spin 2 in a five dimensional space and the gravitational equations in a four dimensional space. *A. H. Taub* (Princeton, N. J.).

Dugas, René. Sur le choc de deux particules relativistes. C. R. Acad. Sci. Paris **216**, 287–288 (1943). [MF 10049]

Stueckelberg, E. C. G. Un principe qui relie la théorie de relativité et la théorie des quanta. Helvetica Phys. Acta **16**, 173–202 (1943).

MATHEMATICAL PHYSICS

Liebowitz, Benjamin. Development of electromagnetic theory for non-homogeneous spaces. Phys. Rev. (2) **64**, 294–301 (1943). [MF 9515]

When the equations of classical electromagnetic theory are extended to regions in which the specific inductive capacity varies with position, the wave equations contain more than one component of the field vectors. This difficulty can be overcome and separation achieved by using a three dimensional space with metric

$$ds^2 = (1/\epsilon)(dx_1^2 + dx_2^2 + dx_3^2),$$

where ϵ is the specific inductive capacity. However, an additional auxiliary scalar function has to be introduced which is given in terms of $1/\sqrt{\epsilon}$ by a differential equation which is worked out. The wave equations which are derived have essential singularities both at the origin and at infinity for any reasonable choice of $1/\sqrt{\epsilon}$. The simple cases of a point-charge and an electrostatic dipole are discussed in some detail. *G. C. McVittie* (London).

de Broglie, Louis. Sur la propagation de l'énergie lumineuse dans les milieux anisotropes. C. R. Acad. Sci. Paris **215**, 153–156 (1942). [MF 9482]

Bottema, O. The electromagnetic field of a solenoid. Physica **8**, 703–710 (1941). [MF 9540]

The electromagnetic force at any point due to a finite circular solenoid can be expressed by starting with Biot and Savart's law for a small rectangle on the cylinder and then integrating over the cylinder. Usually this is done by first integrating over a transverse (circular) section. The author instead integrates over vertical strips first. In the reviewer's opinion there is no gain in this change of order. The author's final elliptic integral formulation of his results is well known and the discontinuity in passing through the solenoid is the familiar situation of electromagnetic current (or hydrodynamical vortex) sheets. *D. G. Bourgin.*

Buchholz, Herbert. Die Gegeninduktivität koaxialer Kreisringe in Gegenwart eines permeablen Kerns. Z. Techn. Phys. **23**, 221–234 (1942). [MF 9352]

The author deals with the general question of the mutual inductance of circular coaxial rings uniformly spaced around a permeable core. First, an exact expression is derived for the vector potential of a single ring surrounding such a core. From this the solution is obtained for a large number of current rings, valid only for the immediate vicinity of the central plane of symmetry of the windings. In order to put this into more manageable form for computation, the author passes to the limit for infinitely many rings, corresponding to a continuous band of finite width. It is then shown how this result may be adapted for computation in the case of a large, but not infinite, number of rings, by the use of appropriate summation formulas for the remainder term in the expansion. *R. M. Foster* (Brooklyn, N. Y.).

Buchholz, Herbert. Der Hohlleiter von kreisförmigem Querschnitt mit geschichtetem dielektrischem Einsatz. Ann. Physik (5) **43**, 313–368 (1943). [MF 10056]

When a cylindrical wave guide is filled with a dielectric characterized by the complex constant $\epsilon_0(1+i\tan\delta_0)$, the radical γ in $\exp i(\gamma z - \omega t)$ is generally of the form $\alpha_D + i\beta_D$, where α_D does not differ much from the corresponding value α when $\delta = 0$ and the dielectric is free from losses. The quantity β_D is generally positive but, for certain critical values of the wave-length referred to vacuum (ϵ, μ),

$$\lambda_{np}^{(m)} = (2\pi b/x_n)(\epsilon/\epsilon_0)^{\frac{1}{2}}, \quad \lambda_{np}^{(e)} = (2\pi b/x_n')(\epsilon/\epsilon_0)^{\frac{1}{2}},$$

given by the positive roots x_n of $J_p(x) = 0$ and the positive roots x_n' of $J_n'(x') = 0$, it is zero when $\delta_0 = 0$. When δ_0 is not zero there is a minimum value of β_D and even for a very moderate value $\tan\delta_0 = 0.0001$ and $\epsilon_0/\epsilon = 16$ the damping is considerable when the critical vacuum wave-length of the wave guide corresponding to x_1' is 34.1259 cm. with $p = 1$.

The desirability of lowering both the critical wave-length and the damping thus leads to difficulty when the wave guide is filled with a suitable dielectric. A possible way out is to use either an annular filling in contact with the metallic envelope or a cylindrical core. The author therefore considers the general case of a cylindrical core of one material surrounded by an annular core of another material which is in contact on its outer surface with the cylindrical metal envelope. The problem discussed is thus a generalization of that of Hondros and Debye in which waves are propagated in a cylindrical dielectric. Coupled waves of electric and magnetic types are considered and the analysis involves Bessel functions of type $Y_p(aw)$ in addition to the Bessel functions $J_p(aw)$. The complicated transcendental equation is discussed graphically for a constant value 2 of the refractive index and for values from 0 to 1 of the ratio a/b of the radii of the two coaxial cylinders. The first six roots are considered. The characteristics of propagation are illus-

trated by graphs one of which shows the variation, with the wave-length of the impressed oscillation, of the wave-length in the wave guide. Approximate expressions are found for the measure of propagation γ in different cases such as $a/b \ll 1$ and the ϵ_0 for one material small in comparison with that for the other. The damping of axially symmetric waves is discussed and the respective effects of dielectric losses and thermal losses at the wall are compared. The total damping is computed for various cases. With a sufficient approximation it can be found by addition of the two types.

H. Bateman (Pasadena, Calif.).

Müller, Wilhelm. Zur Berechnung der Kräfte an zylindrischen Leitern in ebenen elektrostatischen und elektromagnetischen Feldern. Ann. Physik (5) **42**, 609–633 (1942). [MF 10219]

The author derives a formula for the force and moment exerted by two-dimensional electrostatic fields upon cylindrical conductors, and for the forces on electric currents due to magnetic fields. The methods used are similar to those employed in hydrodynamics and utilize analytic function theory in representing the field and in the evaluation of the integrals connected with these forces. Various illustrations are given, one of them consisting of two circular cylinders in a homogeneous field. The exact solution of the latter problem involves elliptic functions; this solution is also presented, including various representations for it by means of proper series derived from elliptic function theory.

H. Poritsky (Schenectady, N. Y.).

Ledinegg, E. Das Feldlinienbild des dem kreiszylindrischen Hohlraum zugeordneten magnetischen Schwingungstyps. Hochfrequenztech. Elektroak. **62**, 38–44 (1943). [MF 10048]

Since a knowledge of the field lines for vibrations in a cavity is thought to be needed for a study of coupled cavities, the lines of the electric field for a cavity having the form of a circular cylinder are found for a type of vibration in which these lines lie in planes perpendicular to the z-axis (the axis of the cylinder). The lines are found to have the simple equation

$$J_1(\beta r)\cos\phi = C_1 J_1(\beta R), \qquad R = y'_{1,1},$$

where $y'_{1,1}$ is the first positive root of the equation $J_1'(x)=0$. For each value of C_1 there are limits for ϕ as is shown in a diagram which is accompanied by instructions for deriving all the other lines with different C_1's from a single one.

An infinite number of magnetic lines lie on each one of a set of surfaces of rotation round the z-axis and in fact a magnetic line can be regarded as common to such a surface and a cylinder. If $\xi = \beta r$, a cylinder is given by an equation of type $\log(c/\sin\phi) = F_1(\xi)$ and a surface of rotation by an equation of type $\log(c' \sin \pi z/l) = F_2(\xi)$, where

$$F_1(\xi) = \int J_1(\xi)d\xi/\xi^2 J_1'(\xi), \quad F_2(\xi) = \int J_1(\xi)d\xi/J_1'(\xi).$$

A discussion of these integrals is given and in particular $F_2(\xi)$ is represented by a power series in ξ with an additional logarithmic term. It is shown on the basis of a general theorem in differential geometry that for a cavity bounded by a cylindrical surface and plane end plates the E-field is perpendicular to the H-field. The H-cylinders, when regarded as metallically clothed, vibrate with the same frequency as the original cylinder. *H. Bateman.*

Riess, Karlem. Electromagnetic waves in a bent pipe of rectangular cross section. Quart. Appl. Math. **1**, 328–333 (1944). [MF 9909]

The propagation of electro-magnetic waves in a bent pipe of triangular cross section is treated. Maxwell's equations are set up in cylindrical coordinates and the field components are eliminated in favor of the azimuthal components H_θ, E_θ. There results two simultaneous equations for H_θ, E_θ with coefficients which are rational fractional in r, the distance from the axis of the bend. These equations are solved by expanding in powers of the curvature $1/R$. The first approximation corresponds to a straight pipe. The approximation of the next order is determined and the remaining field components are obtained. It is concluded that the propagation of an E or a H wave from a straight pipe into a bent pipe yields reflections which are proportional to $1/R$, and hence intensities which are negligible to a first approximation. According to the author a mixed E and H wave in a straight portion sets up conditions which are not predictable from the results of the paper.

H. Poritsky (Schenectady, N. Y.).

Jouguet, Marc. Sur les courants de Foucault dans un ellipsoïde de révolution. C. R. Acad. Sci. Paris **216**, 523–524 (1943). [MF 10053]

The author studies the currents induced in the solid ellipsoid of revolution of conductibility γ and permeability μ by an uniform alternating magnetic field of frequency ω which is parallel to the axis of revolution. General expressions obtained for the components of the electromagnetic field are series with very complicated general terms, but for $\gamma\omega = 0$ (and for $\gamma\omega = \infty$) they take simple forms. These two particular cases, considered as approximate solutions for low (and high) frequencies and conductibilities, are stressed. *E. Kogbetliantz* (Bethlehem, Pa.).

King, Ronold and Harrison, Charles W., Jr. The distribution of current along a symmetrical center-driven antenna. Proc. I.R.E. **31**, 548–567 (1943). [MF 9217]

The current distribution in a cylindrical center-excited antenna, regarded as a boundary value problem in electromagnetic theory, is formulated by means of an integral equation. The equation obtained is substantially that given by Hallén [Nova Acta Reg. Soc. Sci. Upsaliensis (4) **11**, no. 4, 1–44 (1938)]. A comparable analysis, leading to a somewhat different equation, was obtained by L. V. King [Philos. Trans. Roy. Soc. London **236**, 381–422 (1937)]. The solution of the equation differs only slightly from that carried out by Hallén, the final formulas being expressed in terms of sine and cosine integrals. Sets of curves for the distribution of current are given, covering a wide range of antenna lengths and ratios of length to radius. The conventionally assumed sinusoidal distribution of current is shown to be a fairly good approximation for thin antennas, and for thicker antennas which do not greatly exceed one-half wave length in total length. *R. M. Foster.*

Harrison, Charles W., Jr. The radiation field of long wires, with application to vee antennas. J. Appl. Phys. **14**, 537–544 (1943).

This paper deals with the distribution of current on very long antennas excited at the center and with the radiation field of such antennas. The results obtained are applied to the approximate solution for the radiation field of a long wire resonant vee antenna. The cylindrical center-excited antenna, when analyzed as a boundary value problem, leads

to an integral equation involving the unknown current distribution under the integral sign. The solution of this equation as carried out by Hallén [Nova Acta Reg. Soc. Sci. Upsaliensis (4) 11, no. 4, 1–44 (1938)] is based on a method of successive approximations. The method pursued in this paper is an immediate application of that procedure, the solutions as finally obtained involving a multiplicity of sine and cosine integrals. It should be noted that, in the definition of the cosine integral near the end of the paper, the lower limit of the integral should be infinity and not zero. *R. M. Foster* (Brooklyn, N. Y.).

Graffi, Dario. Sulla impedenza di radiazione delle antenne. Alta Frequenza 12, 3–25 (1943). [MF 10254]

The author gives a simple expression for the impedance of a single antenna as a function of the various forms of its energy. By using the reciprocity theorem he extends this result to two antennae irradiating stationary waves of equal frequency. He shows that the impedance of a nonirradiating antenna is different from that of an irradiating one. He gives also a new proof of the reciprocity theorem.

I. Opatowski (Chicago, Ill.).

King, Ronold. Transmission-line theory and its application. J. Appl. Phys. 14, 577–600 (1943).

Several forms of the solution of the transmission line equation are discussed. These include (1) the usual exponential form, (2) the conventional hyperbolic solution and (3) a third form of solution offering many advantages, which is called the completely hyperbolic form. This differs from the conventional hyperbolic solution by expressing the terminal impedances in terms of hyperbolic functions. Formulas for the input resistance, reactance, conductance and susceptance for a section of line of any length and terminated in an arbitrary impedance are given in terms of this generalized solution. Various curves and circle diagrams illustrate the formulas. In the second part of the paper these formulas are applied to several practical problems and, in the final section, very briefly to wave guides.

R. M. Foster (Brooklyn, N. Y.).

Brown, B. M. Application of operational methods to switching problems. Math. Gaz. 26, 157–160 (1942). [MF 10128]

The application of operational methods to switching problems in electric circuit theory is discussed. Two specific examples are worked out in detail. *A. E. Heins.*

Chapman, Sydney. A theoretical note on the magnetic field of a circular sunspot. Terr. Magnetism 49, 37–42 (1944). [MF 10232]

It is assumed that a pair of associated sunspots of opposite magnetic polarities represent two ends of an electromagnet connecting them beneath the sun's surface. The author calculates the magnetic field H at the ends of a semicircular electromagnet in the form of a half anchor-ring and compares the result obtained with observed magnetic fields over the sunspots. These fields vanish at the circular edge of the spot but, as the author shows, the semi-thoroidal model of two associated sunspots studied in this paper cannot verify this condition. *E. Kogbetliantz* (Bethlehem, Pa.).

Waldmann, L. Der Diffusionsthermoeffekt. Z. Phys. 121, 501–522 (1943). [MF 9976]

A thermal flux of diffusion is mentioned by Chapman and Enskog. To develop the theory the author considers atomic masses m, M and distribution functions f, F in a binary mixture of gases so that numbers $dn=fdvdw$ and $dN=Fdvdw$ of atoms of the two kinds are in the volume element dv around the point (x, y, z) and in the velocity element $dw=dc\,dc\,dc$ around the velocity point c, c, c in the hodograph space. In a collision the variation with time of f and F is given by equations of type

$$(1)\qquad \begin{aligned} \partial f/\partial t+c\ \text{grad}\ f&=B(f,f)+B(f,F),\\ \partial F/\partial t+c\ \text{grad}\ F&=B(F,F)+B(F,f) \end{aligned}$$

given by Boltzmann. It is now assumed that in stationary conditions the concentration and temperature T depend only on x. If the relative concentrations are $\gamma=n/(n+N)$ and $1-\gamma=N/(n+N)$, expressions of type

$$(2)\qquad \begin{aligned} f&=g(1+r\partial\gamma/\partial x+s\partial L/\partial x),\\ F&=G(1-R\partial\gamma/\partial x+S\partial L/\partial x), \end{aligned} \qquad L=\log T,$$

$$(3)\qquad \begin{aligned} g&=n(m/H)^{\frac{1}{2}}\exp(-\epsilon/kT),\\ G&=N(M/H)^{\frac{1}{2}}\exp(-E/kT), \end{aligned} \qquad H=2\pi kT,$$
$$\epsilon=\tfrac{1}{2}m(c_x^2+c_y^2+c_z^2),\quad E=\tfrac{1}{2}M(c_x^2+c_y^2+c_z^2),$$

must be expected for f and F in terms of the distribution functions g and G for thermodynamic equilibrium. The coefficients r, s, R, S depend on c and are so small that we can put $r\partial\gamma/\partial x\ll 1$, etc. This approximation is used in the substitution of (2) in (1) and use is also made of the relations $B(g, g)=B(g, G)=B(G, g)=B(G, G)=0$. By means of (3) and the supposition that the pressure is constant throughout space, the equation

$$\partial g/\partial x=g\{(1/\gamma)\partial\gamma/\partial x+[(\epsilon/kT)-(5/2)]\partial L/\partial x\}$$

is obtained and a similar equation is found for $\partial G/\partial x$. Relations which determine r, s, R and S are then found. The diffusion and energy currents are next considered and a relation is found between thermal diffusion and the coefficient of thermal flux of diffusion.

To get equations for diffusion and heat conduction use is made of Enskog's equation for the heat current

$$q=-\lambda\partial T/\partial x-\alpha pD\partial\gamma/\partial x+(5p/2)u_T,$$

in which the first two terms depend on the gradients of temperature and concentration, respectively, while the third represents the transportation of enthalpy by the total flow of particles. Use is made also of the equations of conservation of the number of particles in the stationary state and the final equations are, if $L=\log T$,

$$\begin{aligned} 0&=\partial\gamma/\partial t-\partial/\partial x\{D[\partial\gamma/\partial x+\alpha\gamma(1-\gamma)\partial L/\partial x]\}\\ &\quad+D(\partial L/\partial x)[\partial\gamma/\partial x+\gamma(1-\gamma)\partial L/\partial x]+u_T\partial\gamma/\partial x,\\ (5/2)(n+N)k\partial T/\partial t&=dp/dt+\partial/\partial x(\lambda\partial T/\partial x+\alpha pD\partial\gamma/\partial x). \end{aligned}$$

On the assumption that the differences in temperature in a gas mixture arise only from the diffusion current the first of these equations is replaced by $\partial\gamma/\partial t=\partial/\partial x(D\partial\gamma/\partial x)$. The two basic equations are next written in the vector form

$$\begin{aligned} \partial\gamma/\partial t&=\text{div}\ (D\ \text{grad}\ \gamma),\\ (c_v/v)\partial T/\partial t-dp/dt&=\text{div}\ [(\lambda\ \text{grad}\ T)+\alpha pD\ \text{grad}\ \gamma], \end{aligned}$$

suitable for three dimensions. Some special boundary problems are then solved, the basic equations in the first case being

$$\partial\gamma/\partial t=D\partial^2\gamma/\partial x^2,\quad \partial T/\partial t=K\partial^2T/\partial x^2+\delta T\partial\gamma/\partial t^2$$

and the initial conditions being $T=T_0$ for $0<x<1$, $\gamma=\gamma_0$, $-1<x<0$, $\gamma=\gamma_\infty-(\gamma_0-\gamma_\infty)$ with the supplementary conditions $T=T_0$ for $x=0$, $\gamma=\gamma_\infty$, $\partial\gamma/\partial x=\partial T/\partial x=0$ for $x=\pm 1$. Two expressions are given for the variation of T and graphs are given to illustrate the special case $D=K$. The temperature time integral is found and some numerical

examples are worked out. A discussion of some possible ways of using the effect is given at the end. *H. Bateman.*

Mataré, H. F. Brownsche Bewegung und Widerstandrauschen. Ann. Physik (5) **43**, 271–278 (1943). [MF 9924]

The Nyquist formula for thermal voltage across a resistance $\bar{e}^2 = 4kTRd\nu$ is developed by the method of G. L. de Haas Lorentz from the Brownian motion of electrons, and also from a general oscillation equation of Einstein.

C. E. Shannon (New York, N. Y.).

Meixner, J. Zur Thermodynamik der irreversiblen Prozesse in Gasen mit chemisch reagierenden, dissoziierenden und anregbaren Komponenten. Ann. Physik (5) **43**, 244–270 (1943). [MF 9923]

This paper is concerned with various results deducible from the equations representing conservation of matter, impulse and energy of a gas mixture when these equations are written in a form sufficiently general to allow for diffusion and change in composition. The author first shows the tensor character of Onsager's relations $a_{ik} = a_{ki}$ by using energy-dissipation as an invariant and then applies this result to transform the total energy-stream $\vec{W}$ with its generalized force system into certain other streams $\vec{W}'$, $\vec{W}''$ with their generalized force systems. He also considers changes in the zero-point of internal energy and their effect upon the equations previously mentioned. It is found that, except for mixtures of noninteracting gases without volume viscosity, only rigid-body motions of gases are reversible. The author considers three possible definitions of thermodynamic equilibrium applicable to irreversible processes: (1) all (generalized) forces vanish; (2) all (generalized) streams vanish; (3) $\vartheta = 0$, where $\vartheta = \rho dS/dt + \operatorname{div} \vec{S}$, ρ = total density, S = entropy per gram. It is shown that (1) and (2) are equivalent to (3). It is proved (for a simple case) that diffusion and transformation of gas components must take place when a thermodynamic system is in (nontrivial) stationary equilibrium. The author shows that, for reacting gases, Onsager's relations follow from the principle of detailed chemical equilibrium, but that the converse relation is generally false. The paper concludes with a combined theory of the diffusion-thermo-effect and the thermo-diffusion-effect. *C. C. Torrance* (Cleveland, Ohio).

Neugebauer, Th. Über die Berechnung der Brechungsindizes von Gasen nach der Bohrschen Quantentheorie und der Quantenmechanik. Monatsh. Math. Phys. **50**, 327–338 (1943). [MF 10485]

It is shown that the classical formula for the index of refraction of gases can be derived from quantum mechanics, while the older quantum theory of Bohr leads to a different, and presumably incorrect, formula. The derivation is carried out both for the diatomic molecule and the asymmetric rotator. A similar situation is pointed out with respect to the calculation of the dielectric constant of gases; quantum mechanics leads to the classical formula, while the Bohr theory does not. *O. Frink* (State College, Pa.).

Petiau, Gérard. Sur une représentation du corpuscule de spin 2. C. R. Acad. Sci. Paris **212**, 47–50 (1941). [MF 9184]

The author shows that the symmetric part of a five dimensional tensor satisfying a first order equation of the type proposed by Proca [J. Phys. Radium (7) **7**, 347–353(1936)] may be interpreted as describing a particle of spin 2.

A. H. Taub (Princeton, N. J.).

Petiau, Gérard. Sur la théorie des corpuscules de spins quelconques. C. R. Acad. Sci. Paris **215**, 77–79 (1942). [MF 9479]

The author proposes an equation of the Dirac type involving a symmetric spinor with n indices and the sum of the direct products of one Dirac matrix with $(n-1)$ unit matrices. He shows how this may be written in Hamiltonian form by introducing a suitable operator. This is formally equivalent to the process of converting p of the indices (where $n = 2p$ or $2p+1$) into those transforming like quantities satisfying the adjoint Dirac equation in contrast to the remaining ones which transform like quantities satisfying Dirac's equation. *A. H. Taub* (Princeton, N. J.).

Petiau, Gérard. Sur les représentations des équations d'ondes des particules à spins entiers. C. R. Acad. Sci. Paris **216**, 438–440 (1943). [MF 10024]

Proca, Alexandre. Sur les propriétés d'une nouvelle particule élémentaire. C. R. Acad. Sci. Paris **216**, 337–339 (1943). [MF 10017]

Tonnelat, Marie-Antoinette. Une nouvelle forme de théorie unitaire. Étude de la particule de spin 2. Ann. Physique (11) **17**, 158–208 (1942). [MF 9227]

A four index spinor is formed by taking products of four single index spinors, two satisfying the Dirac equation and two satisfying the adjoint of the same equation. The equation satisfied by this four index spinor is given and it is shown that it is equivalent to six groups of equations, the first group involving a symmetric tensor (and hence describing a particle of spin 2), the next three groups involving a vector and an anti-symmetric tensor (and hence describing particles of spin one, mesotrons or photons) and finally two groups involving two scalars (and hence describing particles of spin zero). The tensor equations equivalent to the spinor ones are obtained and compared to the gravitational ones as well as those of the mesotrons (and photons). The equations are solved for plane wave solutions and the paper concludes with some remarks on second quantization.

A. H. Taub (Princeton, N. J.).

Courtois, Jacques. Représentation des systèmes de particules identiques à fonction d'onde symétrique. C. R. Acad. Sci. Paris **216**, 35–37 (1943). [MF 10001]

Some elementary generalities concerning the expression of the wave function of a symmetric system of identical particles. The results contain nothing essentially new.

B. O. Koopman (Washington, D. C.).

Courtois, Jacques. Représentation d'un système de particules en nombre indéterminé. C. R. Acad. Sci. Paris **216**, 146–148 (1943). [MF 10007]

Generalities on symmetric systems of identical particles (photons) in which the annihilated photons contribute to the wave function, and the identification of the resulting with ordinary fixed number of symmetric particle functions.

B. O. Koopman (Washington, D. C.).

Ginsburg, V. On the theory of the particle of spin $\frac{3}{2}$. Acad. Sci. USSR J. Phys. **7**, 115–128 (1943). [MF 9967]

The theory for particles of spin $\frac{3}{2}$ found by Pauli and Fierz [Proc. Roy. Soc. London. Ser. A. **173**, 211–232 (1939); these Rev. **1**, 190] is rewritten in spinor-vector notation. The wave equations can be written

$$\gamma^k \partial B_m/\partial x_k + kB_m = 0, \quad \gamma^m B_m = 0,$$

where the spinor indices of B_m and of the Dirac matrices γ^λ are not explicitly written out. The Lagrangian, energy-momentum tensor, eigenfunctions of free motion, current vector, interaction with an external electromagnetic field and magnetic properties of the particles are discussed. The author refers to suggestions of Tamm and Davydov which have not been published. The same formulation of the theory as the author's has been independently found by Schwinger and Rarita [Phys. Rev. (2) 60, 61 (1941) (letter to the editor)]. *W. Pauli* (Princeton, N. J.).

Payne, Mary Hewlett. Interpretation of spinning electron with bipolar coordinates. Phys. Rev. (2) 65, 39–46 (1944). [MF 9880]

It is well known from two dimensional spinor theory that the points of a sphere in three space may be characterized through stereographic projection by a single complex variable and that this complex variable may be considered as the nonhomogeneous coordinate of a point on a complex projective line. Moreover, the group of linear and antilinear transformations of this line into itself is isomorphic to the Lorentz group. This paper derives again some of the known results of two dimensional spinor theory and tries to interpret them in terms of the bipolar coordinates of a point on a sphere, namely, the distances of a point from two fixed diametrically opposite points of the sphere. These are allowed to have complex values. When this is done, they cannot be interpreted in terms of the sphere but must be pictured in terms of the complex projective line, although the author does not seem to take this into account. The latter part of the paper attempts to interpret the spin variable and the Pauli spin operators in terms of these quantities. *A. H. Taub* (Princeton, N. J.).

Lubański, J. K. Sur le spin des particules élémentaires. Physica 8, 44–52 (1941). [MF 9545].

The author uses the definition of the spin operator proposed by Pauli and Fierz [Proc. Roy. Soc. London. Ser. A. 173, 211–232 (1939); these Rev. 1, 190] for a particle of arbitrary spin N which is described by an N-index symmetric spinor satisfying an equation similar to the Dirac equation. The relation is given between the spin so defined and the values of an invariant formed from the operators which relate an infinitesimal Lorentz transformation with its spin image. *A. H. Taub* (Princeton, N. J.).

Lubański, J. K. Sur la théorie des particules élémentaires de spin quelconque. I. Physica 9, 310–324 (1942). [MF 9814]

The author investigates equations

$$(1) \qquad (\Gamma^\lambda \partial/\partial x_\lambda + Nk)\Psi = 0,$$

in which $\lambda = 1, 2, 3, 4$; $x^4 = ict$, N an integer, $\Gamma^\lambda = \sum_{n=1}^{N} \gamma^\lambda_{(n)}$,

$$\gamma^\lambda_{(n)} = I \times \cdots \times \underset{(n)}{\gamma^\lambda} \times \cdots \times \underset{(N)}{I},$$

I unit matrix, $\times$ direct product, $\gamma^\lambda_{(n)}$ Dirac-matrices which commute for different values of n [compare also H. A. Kramers, F. J. Belinfante, J. K. Lubański, Physica 8, 597 (1941)]. The wave function Ψ has 4^N components. First the irreducible representations of the ring $\{\Gamma\}$ are determined using the fact that the ten matrices $(i/2)\Gamma^\lambda$, $\Gamma^\lambda\Gamma^\mu - \Gamma^\mu\Gamma^\lambda$ are operators of infinitesimal rotation in a five-dimensional space. These irreducible representations $D_{(r,s)}$ are characterized by two integer or half-integer parameters r, s for which $r \geqq s \geqq 0$. The ring $\{\Gamma(N)\}$ consists of the direct product

$$D^{(5)}_{\frac{1}{2},\frac{1}{2}} \times \cdots (N \text{ times}) \cdots \times D^{(5)}_{\frac{1}{2},\frac{1}{2}}$$

which decomposes in the different $D^{(5)}_{(r,s)}$ with $r \leqq N/2$, $s \leqq N/2$ and r, s integer (half-integer) if N is even (odd). The rest-masses of the particles described by the equation (1) with $\Gamma^\lambda_{(r,s)}$ corresponding to the irreducible representation $D^{(5)}_{(r,s)}$ are given by

$$M_\rho = \frac{\hbar\kappa}{c}\frac{N}{2\rho}, \qquad \rho = r, r-1, \cdots.$$

The solutions of the wave equations which correspond to a given value of the rest mass (that means to given values of r, s, ρ) again decompose according to different values of the spin σ which is defined as the result of the operator

$$S^2 = \{\tfrac{1}{2}T^{\lambda\mu}T_{\lambda\nu}p_\mu p^\nu - T^{\lambda\mu}T_{\lambda\nu}p_\mu p^\nu\}/p_\rho p^\rho,$$

where $T^{\lambda\mu}$ is the operator of the total angular momentum. For the number $Z(\rho, \sigma)$ of the independent solutions of (1) corresponding to given values r, s of the parameters of the irreducible representation and to given values of rest mass and spin, the author gives the formula

$$(2) \quad Z(\rho,\sigma) = \min\,([\tfrac{1}{2}\{(r-|\rho|)-|(s-\sigma)|\}+1], [\sigma-\epsilon+1], [s-\epsilon+1], r-\sigma+1, r-s+1);$$

$Z(\rho, \sigma) = 0$ if this formula gives a negative value; in this formula $[x]$ is the largest integer smaller than or equal to x and

$$\epsilon = \tfrac{1}{2}\{(r-\rho)-(s-\sigma)\} - [\tfrac{1}{2}\{(r-\rho)-(s-\sigma)\}] = \begin{cases} \frac{1}{2} \\ 0 \end{cases}.$$

The author emphasizes that the energy of this particle is partly negative and that the equations cannot be written consistently in a simple way if an external electromagnetic field is present. The relation of this theory to the theory of Fierz and Pauli [Proc. Roy. Soc. London. Ser. A. 173, 211–232 (1939); these Rev. 1, 190] is discussed.

W. Pauli (Princeton, N. J.).

Lubański, J. K. Sur la théorie des particules élémentaires de spin quelconque. II. Physica 9, 325–338 (1942). [MF 9815]

Some properties of the irreducible representations of the group of relations in a five-dimensional space which have been used in the paper reviewed above are derived and the proof of the formula (2) for $Z(\rho, \sigma)$ is given.

W. Pauli (Princeton, N. J.).

Lubański, J. K. et Rosenfeld, L. Sur la représentation des champs mésiques dans l'espece à cinq dimensions. Physica 9, 117–134 (1942). [MF 9811]

The algebra defined by the Duffin-Kemmer relations

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\mu\beta_\nu\beta_\rho = \beta_\mu\delta_{\nu\rho} + \beta_\rho\delta_{\nu\mu}$$

for five dimensions (indices varying from 1 to 5) is investigated. There exist five irreducible representations with the degrees 1, 6, 15, 10, 10, respectively. The equations

$$\beta_\mu\partial\psi/\partial x_\mu + \kappa\psi = 0$$

are discussed especially with respect to applications to mesons if the additional assumption is made that ψ is independent of the fifth coordinate. *W. Pauli.*

Pais, A. Meson fields in projective space. Physica 9, 267–284 (1942). [MF 9813]

At present the meson field theory which combines a vector and a pseudoscalar field with equal coupling con-

stants seems to be of greatest physical interest; therefore the author investigates the possibility of reducing the number of arbitrary constants by the consideration of a wider group of transformations than the Lorentz group. Möller has investigated the rotation in the five dimensional de Sitter space and found that it does not yield significant physical consequences. Hence present work is devoted to the examination of meson field theories from the point of view of projective relativity. Lubański and Rosenfeld [cf. the preceding review] have shown that, for meson fields in five dimensional space, there exist five unequivalent irreducible representations of degrees 1, 6, 10, 10 and 15. The first is a trivial one, the second corresponds to a scalar theory and the third and the fourth, though algebraically different, are physically equivalent and correspond to the pseudovector theory. The fifth corresponds to the 5-vector theory and is of greatest physical interest. It is shown that, for a free meson field, the last representation leads to a great number of basic types which are in accordance with the projective conceptions, namely, any of the Kemmer types obtained from consideration of the Lorentz transformation alone, and any combination of a vector or pseudovector with a scalar or pseudoscalar. If for each of the representations the corresponding expressions for the nuclear source densities which can be obtained by means of the Dirac matrices alone are considered, the three different representations lead to the scalar, pseudovector and the mixed theory with reduced number of constants, respectively. However, if a projective 5-vector is introduced in the source function, it does not present anything new in the representations of degrees 6 and 10, but in the representation of degree 15 it yields two more basic types, a vector and a pseudoscalar. Thus in the projective treatment the basic types of field are the four Kemmer cases and in addition the mixed theory which occupies as fundamental a position as the others. *S. Kusaka.*

Pais, A. On the electric charge current density of a nuclear system. Physica 9, 407–421 (1942). [MF 9817]

The author considers a system of nuclear particles which interact with short range forces according to the mixed meson field theory. He computes the energy momentum tensor of the system using the formalism of five dimensional projective space [see the preceding review]. From this tensor the Hamiltonian and the electric charge current density are derived. The Hamiltonian is transformed by separating successively the longitudinal part of the electromagnetic field and the static meson field, and the corresponding transformation of the electric charge current density is discussed in detail. From these results the electric dipole and quadrupole moments and the magnetic dipole moment of the nuclear system are computed to terms of the second order in the coupling constants. *S. Kusaka.*

Born, Max and Peng, H. W. Quantum mechanics of fields. I. Pure fields. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 40–57 (1944). [MF 10130]

It is shown that in a general family of field theories there exist particular solutions of the field equations in which the densities of energy and momentum are independent of space and time. To these field theories belong all linear and nonlinear gauge invariant theories, which may be, for instance, scalar, vector or spinor theories. The particular solutions in question turn out to be plane waves where the field quantities have a phase factor $\exp i[k\cdot x - k_0 x_0]$ or its complex conjugate. The authors show how the quantizations of such a particular solution, characterized by a given value of the four vector (k, k_0), lead to a problem of ordinary wave-mechanics and can be performed in an elegant way by using the mentioned independence of the energy-momentum densities on space-time as the starting point. [In the reviewer's opinion section 4 on a real field needs an essential amendment.] The quantization of the general solutions of the field equations, where in a nonlinear theory the particular solutions corresponding to different values of k are intermingled in a complicated way, is not considered in this paper. *W. Pauli* (Princeton, N. J.).

McConnell, James. Non-linear quantum electrodynamics of the vacuum. Proc. Roy. Irish Acad. Sect. A. 49, 149–176 (1943). [MF 9986]

The author studies the quantum mechanics of an electromagnetic field, based on Born's nonlinear electrodynamics. The topics considered include the refraction of light by light and the mutual scattering of two light rays. The latter problem is first treated by resolving the field into circularly polarized waves, but plane polarized rays are also considered. By means of a properly chosen Lorenz frame the case of any two rays is reduced to a pair of oppositely directed rays of equal frequency. The usual linear field is described by means of a quadratic Hamiltonian whose canonical variables are the amplitudes of the oscillations in a parallelepiped. The nonlinear theory is introduced by the addition of fourth degree terms to the Hamiltonian, and making these relativistically invariant. The effect of radiation damping on the scattering process is also studied. The author states that the results obtained are too small to be verified by theory. *H. Poritsky.*

König, H. W. Über das Verhalten von Elektronenströmen im elektrischen Längsfeld. Hochfrequenztech. Elektroak. 62, 76–86 (1943). [MF 10131]

This paper studies plane electron beams in a one dimensional longitudinal electric field, the latter being variable with time. The equations are reduced to a form similar to the hydrodynamic equations and the variables are reduced to dimensionless form. The case of a sinusoidal current is investigated. General integrals are obtained by means of the characteristics of the resulting partial differential equations. A solution of the time of transit is obtained by means of the inversion theorem of Lagrange. *H. Poritsky.*

BIBLIOGRAPHICAL NOTE

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